CO-ORDINATE GEOMETRY

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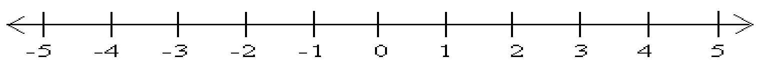
7.1 Cartesian Co-ordinate System

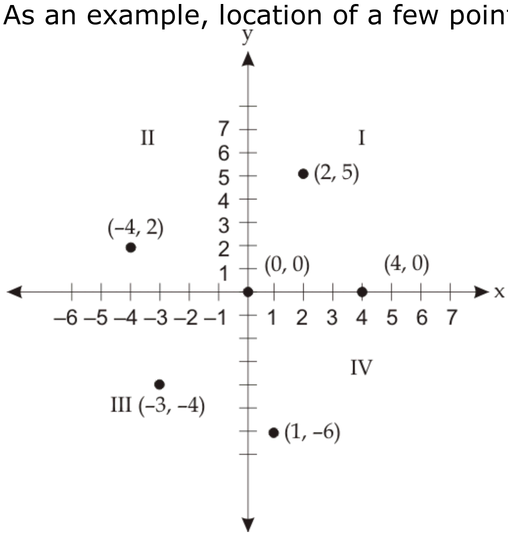
All real numbers can be precisely represented on a number line. 0 serves as the origin or the reference point for the number line. Thus, all numbers to the right of 0 are positive and all numbers to its left are designated as negative. Further, if a number is to the right of a second number then it is greater in value than the second number.



For showing the position of a point in a two dimensional plane, two co-ordinate axis, which are perpendicular to each other, are used. The horizontal axis is called the x-axis and the vertical axis is called the y-axis. The address of any point is of the form (x,y) where the first number x corresponds to location on x-axis and the second number y corresponds to location on the y-axis. Together these two give a unique address to each point in a plane.
The two axis divide the plane into 4 sections which are called Quadrants I, II, III and IV. The x and y co-ordinates have a + or – sign depending on in which quadrant a point is located. The x and y co-ordinates of a point in the quadrants are:

Quadrant	x-co-ordinate	y-co-ordinate	Coordinates
I	+	+	(+, +)
II	-	+	(-, +)
III	-	-	(-, -)
IV	+	-	(+, -)

As an example, location of a few points is shown below.



Note:

- The co-ordinates of the origin are (0, 0).
- The x co-ordinate of every point on the y-axis is zero.
- The y co-ordinate of every point on the x-axis is zero.

7.2 Distance formula

The distance between two points (x_1, y_1) and (x_2, y_2) $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Example: Find the distance between the points A (-3, 2) and B (-4, -3).
Solution: $AB=\sqrt{(-3+4)^2+(2+3)^2}$ = $\sqrt{26}$ units.

Example: What kind of triangle is ∆ PQR if P(12, 8), Q(- 2, 6) and R(6, 0)?
Solution: By distance formula,

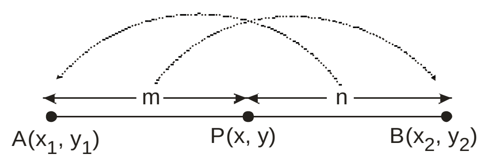
PQ = $\sqrt{(12+2)^2+(8-6)^2} = \sqrt{200} = 10\sqrt{2}$

QR = $\sqrt{(-2-6)^2+(6-0)^2} = \sqrt{100} = 10$

PR = $\sqrt{(12-6)^2+(8-0)^2} = \sqrt{100} = 10$

Since QR=PR and $QR^2+PR^2=PQ^2$, ∆ PQR is an isosceles right triangle.

7.3 Section formula



If a point P divides the line segment joining two points A(x_1, y_1) and B(x_2, y_2) internally in the ratio m:n (i.e., PA:PB = m:n), then the co-ordinates of P are given by:

$$P(x,y)=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right).$$

If P divides the line segment AB externally in the ratio m:n (i.e., P lies on the extended line segment AB) then: $P(x,y)=\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right)$.

Example: Find the co-ordinates of the point which divides the join of A (-2,1) and B (2,4) (i) internally (ii) externally in the ratio 3:2.

Solution: For internal division, $P(x,y)=\left(\frac{3\times2+2\times(-2)}{3+2},\frac{3\times4+2\times1}{3+2}\right)=(\frac{2}{5},\frac{14}{5})$

For external division, $P(x,y)=\left(\frac{3\times2-2\times(-2)}{3-2},\frac{3\times4-2\times1}{3-2}\right)=(10,10)$

Example: Find the ratio in which x-axis divides the segment joining A (-3, -2) and B (4, -4). Also find the point of division. Is the division internal or external?

Solution: If P(x, 0) is the point of division and if the ratio is k:1, then,

x = $\frac{k\ X\ 4+1(-3)}{k+1}$ and $0=\frac{k(-4)+1(-2)}{k+1}$

x = $\frac{4k-3}{k+1}$ and $0=\frac{-4k-2}{k+1}$

∴ -4k – 2 = 0 or k = $-\frac{1}{2}$

So the ratio is k:1 = $-\frac{1}{2} : 1$ = -1:2

The negative sign implies that the division is external.

x = $\frac{4\left(-\frac{1}{2}\right)-3}{-\frac{1}{2}+1}$ = - 10 ∴The point of division is (-10, 0).

As a special case of section formula, the co-ordinates of midpoint (P) of a line segment AB are given by: P(x, y) = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

7.4 Application to geometrical figures

If A(x_1,y_1), B(x_2,y_2) and C(x_3,y_3) are the vertices of a triangle, then:

- co-ordinates of the Centroid : $G(x, y) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$
- Co-ordinates of the In-centre: $I(x, y) = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$ Where a, b and c are the lengths of three sides opposite to ∠A, ∠B and ∠C respectively.
- Area of the triangle $= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}[x_1(y_2-y_3) - x_2(y_1-y_3) + x_3(y_1-y_2)]$

Note: If the area of a triangle = 0, then the 3 points are collinear.

Example: Find the area of the triangle whose vertices are (3,5), (2, 0) and (6, 1).

Solution: Here, (x_1, y_1) = (3,5), (x_2, y_2) = (2, 0), (x_3, y_3) = (6,1)

Area of the triangle $= \frac{1}{2}[x_1(y_2-y_3) - x_2(y_1-y_3) + x_3(y_1-y_2)]$

$= \frac{1}{2}\left|[3(0-1)\right) - 2(5-1) + 6(5-0)]| = \frac{1}{2}|[-3-8+30]| = \frac{19}{2} = 9.5\, sq.units.$

- **Condition for a parallelogram:** If A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) and D(x_4, y_4) are four non-collinear points such that $x_1+x_3=x_2+x_4$ and $y_1+y_3 = y_2+y_4$, then □ABCD is a parallelogram.

Example: In a parallelogram ABCD, where A (3,5), B (2, 0), C (6, 1), find the co-ordinates of D.

Solution: Let the co-ordinates of D be (x_4, y_4).

Since □ABCD is a parallelogram, $x_1+x_3 = x_2+x_4$

∴ 3+6 = 2+x_4 ∴ x_4 = 7

Also, $y_1+y_3 = y_2+y_4$ ∴ 5+1 = 0+y_4 ∴ y_4 = 6

∴The co-ordinates of D are (7, 6).

7.5 Straight Line

The **inclination** of a straight line is defined as the measure of the angle, which it makes with the positive direction of x-axis (or any line parallel to it), measured in the anticlockwise direction.

Slope (or Gradient) (denoted by 'm') of a straight line **i**s defined to be tan θ, where θ is the inclination of the line. Thus for:

- a line parallel to x-axis: m = 0
- a line perpendicular to x-axis: m = not defined
- a line making an acute angle θ (with x-axis) : m > 0
- a line making an obtuse angle θ: m < 0
- two lines with slopes m_1 and m_2 to be parallel: $m_1 = m_2$
- two lines with slopes m_1 and m_2 to be perpendicular to each other: $m_1 \times m_2 = -1$.
- Slope of a line segment formed by joining two points (x_1, y_1) and (x_2, y_2): $m = \frac{y_1 - y_2}{x_1 - x_2}$

7.5.1 Different forms of Equation of a straight line

(i) **Point Slope form**: The equation of a straight line passing through the point (x_1, y_1) and having slope m is: $y - y_1 = m(x - x_1)$.

(ii) **Two Point form**: The equation of a straight line passing through the points (x_1, y_1) and (x_2, y_2), is: $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, where $x_1 \neq x_2$.

(iii) **Slope Intercept form**: The equation of a line having slope m and making an intercept b on y-axis is $y = mx + b$.

(iv) **Double Intercept form**: The equation of a line making intercepts a and b on x and y axis respectively is: $\frac{x}{a} + \frac{y}{b} = 1$. where a, b ≠ 0 .

(v) **Normal form**: If the perpendicular drawn from the origin to a line has inclination α and length p, then the equation of the line is $x \cos \alpha + y \sin \alpha = p$.

(vi) **Symmetric form** : The equation of a line passing through a point (x_1, y_1) and making an angle of θ with x-axis is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$

(vii) **General form:** $Ax + By + C = 0$, where A, B and C are real numbers.

Notes:

- The different forms of equation of a straight line can be easily converted into each other and the slope and intercepts can be worked out. For example, in case of General form: Slope m = $-\frac{A}{B}$, x intercept = $-\frac{C}{A}$, Y intercept = $-\frac{C}{B}$
- $Ax + By + K = 0$ represents a line parallel to $Ax + By + C = 0$
- $Bx - Ay + K = 0$ represents a line perpendicular to $Ax + By + C = 0$.
- The x intercept of a line can be obtained by substituting y = 0 and the y intercept can be obtained by substituting x = 0 in the equation of the line.
- Equation of a line parallel to x axis will be of the form: y=k, where k is a constant representing y-intercept.
- Equation of a line parallel to y axis will be of the form: x=k, where k is a constant representing x-intercept.
- The point of intersection of any two lines can be found by solving both the equations for x and y simultaneously.

Example: Write the equation 3x+4y-8=0 in the slope intercept form and find the points where the line cuts x and y axis.

Solution: Re-arranging the equation we get: $y = \frac{-3}{4}x + 2$

On comparing with slope intercept form we find: slope m= $\frac{-3}{4}$ and b =2. Hence the point of y-intercept is (0,2). For x-intercept, put y=0 and on solving we get x= $\frac{8}{3}$. Hence the point is = $(\frac{8}{3}, 0)$

Example: Find the equation of the line which passes through (-1,3) and which is perpendicular to the line passing through (2,3) and (4, 7).

Solution: Slope of the given line = $\frac{7-3}{4-2} = 2$ ∴ The slope of the required line will be = $\frac{-1}{2}$

Since the line passes through (-1, 3). equation of the line is:
y – 3 = $\frac{-1}{2}$ (x + 1) or x+2y-5=0

Example: Find the co-ordinates of the foot of the perpendicular from the point (-1, 2) on the line 2x – 3y + 7 = 0.

Solution: Let co-ordinates of the foot of the perpendicular be (a, b)
Since it lies on the given line, ∴ 2a – 3b + 7 = 0 ...(i)
Slope of the given line $= \frac{2}{3}$ ∴Slope of perpendicular $= -\frac{3}{2}$

$\therefore \frac{b-2}{a+1} = -\frac{3}{2}$ ⇒ 3a+2b-1 = 0 ...(ii)

Solving (i) and (ii), a = $\frac{-11}{13}$ and b = $\frac{23}{13}$

Hence the foot of the perpendicular is $(\frac{-11}{13}, \frac{23}{13})$.

Example: Find the equation of the locus of a point P equidistant from the points A(-3, 2) and B(5,4).

Solution: Let the co-ordinates of P be (x, y). PA = PB

$\therefore \sqrt{(x+3)^2 + (y-2)^2} = \sqrt{(x-5)^2 + (y-4)^2}$ ⇒ 4x + y = 7

Here locus represents the equation of perpendicular bisector of the two points.

7.5.2 Angle between two lines

- If m_1 and m_2 are the slopes of two lines such that $m_1 \times m_2 \neq -1$, then:
  tan θ = $\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, where θ is the acute angle between the two lines.

- If the equation of the two lines are $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ then, tan $\theta = \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}$

Example: Find the equation of the straight lines passing through the point (2, 3) and inclined at an angle of 45° to the line 3x + y – 5 = 0.

Solution: Slope of the given line, $m_1 = -3$

If m_2 is the slope of the required line, $\tan 45° = \left|\frac{-3-m_2}{1-3m_2}\right| = 1$

$\therefore \frac{-3-m_2}{1-3m_2} = 1 \ \ or \ \ \frac{-3-m_2}{1-3m_2} = -1 \quad \Rightarrow \quad m_2 = -\frac{1}{2}$ or 2.

Since the point (2, 3) lies on the line, the equations of the line is:

$y - 3 = -\frac{1}{2}(x - 2)$ and $y - 3 = 2(x - 2)$

$\therefore$ x + 2y – 8 = 0 and 2x – y – 1 = 0 are the required equations.

7.5.3 Distance from a line

- If $P(x_1, y_1)$ is any point and Ax + By + C = 0 is a line, then the perpendicular distance of P from the line is given by $\left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right|$.

  If $Ax_1 + By_1 + C$ is positive, the point $P(x_1, y_1)$ lies on the origin side of the line and if $Ax_1 + By_1 + C$ is negative, then $P(x_1, y_1)$ lies on the non-origin side of the line.

- If $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ are two parallel lines, the perpendicular distance between them is given by $\left|\frac{C_1 - C_2}{\sqrt{A^2 + B^2}}\right|$.

Example: Find the distance between the lines 4x + 3y – 5 = 0 and 8x + 6y + 15 = 0.

Solution: Making coefficients of x and y equal,

$4x + 3y - 5 = 0$ ...(i)

$4x + 3y + \frac{15}{2} = 0$ ...(ii)

$\therefore$ Distance between the parallel lines $= \left|\frac{C_1 - C_2}{\sqrt{A^2 + B^2}}\right| = \left|\frac{\frac{15}{2} - (-5)}{\sqrt{4^2 + 3^2}}\right| = 2.5$ units

7.6 Equation of a circle

If the centre of the circle is C(h, k) and radius is r, the equation of the circle is given by:

$\sqrt{(x-h)^2 + (y-k)^2} = r$.

Example: Find the centre and radius of the circle represented by the equation $4x^2+4y^2-24x+16y+27=0$

Solution: $4x^2+4y^2-24x+16y+27=0$

$4x^2$-24x+4y^2+16y = -27
4(x^2-6x)+4(y^2+4y) = -27
$4(x-3)^2+4(y+2)^2 = 25$
$(x-3)^2+(y+2)^2 = (\frac{5}{2})^2$
Hence, centre of circle is (3,-2) and radius is 2.5

Example: One end of a diameter of a circle with centre C(2, 3) is P(7, 6). Find the co-ordinates of the other end of the diameter.

Solution: Let Q(x, y) be the other end of the diameter. Since C is the midpoint of PQ,

$2 = \frac{x+7}{2}$ and $3 = \frac{y+6}{2}$

On solving we get, x = -3 and y = 0. ∴Co-ordinates of Q are (-3, 0).