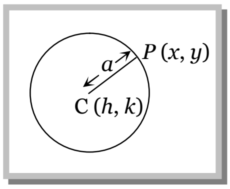
Circle and System of Circles

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always constant. A fixed point is called the centre of the circle and the distance is called the radius of the circle.

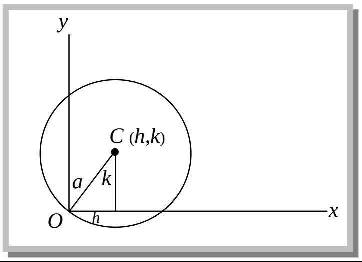
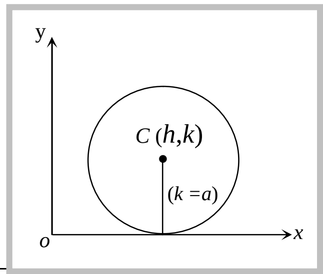
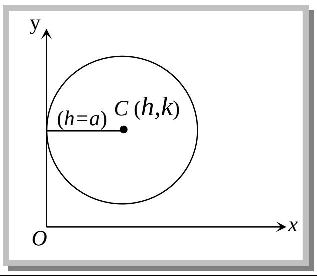
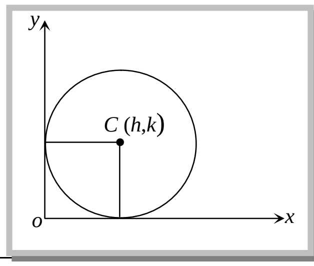
Different form of equations of circles

(1) **Central form :** Let C be the centre of the circle and its coordinates be (h, k). Let the radius of the circle be a and let $P\ (x, y)$ be any point on the circumference. Then, $CP = a \Rightarrow CP^2 = a^2 \Rightarrow$ $(x-h)^2 + (y-k)^2 = a^2$.



(2) **Standard form :** The equation of the circle with centre at (0, 0) and radius 'a' is $x^2 + y^2 = a^2$.

(3) **Special cases :**

(a) **Where circle passes through the origin :** In this case $a^2 = h^2 + k^2$ $\therefore$ Equation of circle is $x^2 + y^2 - 2hx - 2ky = 0$.  	(b) **When the circle touches x-axis :** In this case $k = a$. $\therefore$ Equation of circle is; $x^2 + y^2 - 2hx - 2ay + h^2 = 0$. 
(c) **When the circle touches y- axis :** In this case $h = a$. $\therefore$ Equation is $x^2 + y^2 - 2ax - 2yk + k^2 = 0$.  	(d) **When the circle touches both axes:** In this case $h = k = a$ $\therefore$ Equation is $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. 

(4) When the circle passes through the origin and centre lies on x-axis, then $k = 0$ and $h = a$. Therefore equation is $(x-a)^2 + (y-0)^2 = a^2$ or $x^2 + y^2 + 2ax = 0$.

(5) When the circle passes through the origin and centre lies on y-axis, then $h = 0$ and $k = a$. Therefore equation is $x^2 + y^2 - 2ay = 0$.

(6) **General equation of a circle :** The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle whose centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

Note : ❑ The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle of radius $\sqrt{g^2 + f^2 - c}$.

If $g^2 + f^2 - c > 0$, then the radius of the circle is real and hence the circle is also real.

If $g^2 + f^2 - c = 0$, then the radius of the circle is zero. Such a circle is known as a point circle.

If $g^2 + f^2 - c < 0$, then the radius $\sqrt{g^2+f^2-c}$ of the circle is imaginary but the centre is real. Such a circle is called an imaginary circle as it is not possible to draw such a circle.

Note : ❑ Special features of the general equation $x^2+y^2+2gx+2fy+c=0$ of the circle are:

(a) It is quadratic in both *x* and *y*.

(b) Coefficient of x^2 = Coefficient of y^2. In solving problems it is advisable to keep the coefficient of x^2 and y^2 unity.

(c) There is no term containing *xy* *i.e.* the coefficient of *xy* is zero.

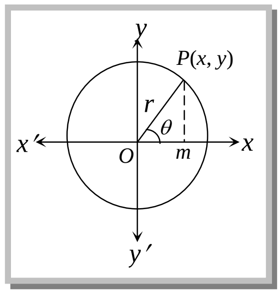
(d) It contains three arbitary constants viz, *g*, *f* and *c*.

Note : ❑ The equation $ax^2+ay^2+2gx+2fy+c=0$, $a \neq 0$ also represents a circle. This equation can also be written as $x^2+y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0$. The coordinates of the centre are $\left(-\frac{g}{a},-\frac{f}{a}\right)$ and radius $=\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a^2}}$.

Note : ❑ On comparing the general equation $x^2+y^2+2gx+2fy+c=0$ of a circle with the general equation of second degree $ax^2+2hy+by^2+2gx+2fy+c=0$ we find that it represents a circle if $a=b$ *i.e.*, coefficient of x^2 = coefficient of y^2 and $h=0$ *i.e.*, coefficient of *xy* = 0.

(7) **Equation of a circle when the co–ordinates of end points of a diameter are given :** The equation of the circle drawn on the straight line joining two given points (x_1,y_1) and (x_2,y_2) as diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

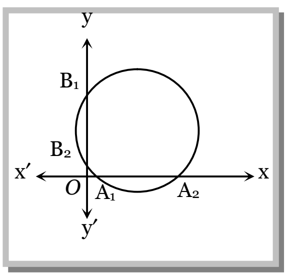
(8) **Equation of a circle in parametric from :** Parametric equation of $x^2+y^2=r^2$: Let $P(x,y)$ be any point on the circle $x^2+y^2=r^2$ and let OP make an angle θ with the positive direction of *x* – axis, Let *PM* be perpendicular from *P* on *x –axis*. Then $\angle MOP=\theta$ $\therefore x=OM=r\cos\theta$ and $y=PM=r\sin\theta$. Thus, $x=r\cos\theta$ and $y=r\sin\theta$ are the required parametric form of the equations $x^2+y^2=r^2$, where θ is a parameter and $x=h+r\cos\theta$, $y=k+r\sin\theta$ are the required parametric form of the circle $(x-h)^2+(y-k)^2=r^2$.



(9) **Three point form :** There is one and only one circle through three points not on the same line. If the points are $(x_1,y_1),(x_2,y_2),(x_3,y_3)$, then the equation of the circle is : $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$.

Intercepts on the axes

The length of intercepts made by the circle $x^2+y^2+2gx+2fy+c=0$ with *x* and *y*-axis are $2\sqrt{g^2-c}$ and $2\sqrt{f^2-c}$ respectively.



Note : ❑ If $g^2 > c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and distinct, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the *x*-axis in two real and distinct points and the length of the intercept on *x*-axis is $2\sqrt{g^2 - c}$. If $g^2 = c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and equal, so the circle touches *x*-axis and the intercept on *x*-axis is zero.

If $g^2 < c$, then the roots of the equations $x^2 + 2gx + c = 0$ are imaginary, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not meet x- axis in real points.

Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the *y*-axis in real and distinct points, touches or does not meet in real points according as $f^2 >, = \text{or} < c$.

Position

(1) **Of a point with respect to a circle :** The point $P(x_1, y_1)$ lies out side, on or inside a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, = \text{or} < 0$.

(2) **Of a line with respect to a circle :** The line $y = mx + c$ does not intersect or touches or intersect a circle $S = x^2 + y^2 - a^2 = 0$ according as, $c^2 > a^2(1 + m^2)$ or $= a^2(1 + m^2)$ or $< a^2(1 + m^2)$ respectively.

The length of the intercepts cut off from a line by a circle

The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2(1 + m^2) - c^2}{(1 + m^2)}}$.

Equation of tangent of circle

(1) Equation of the tangent to the given circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) on it, is given by $xx_1 + yy_1 = a^2$.

(2) Equation of the tangent to the given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at point (x_1, y_1) on it, is given by $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

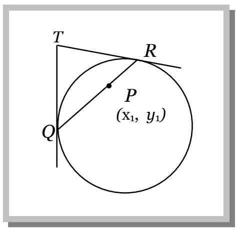
(3) The straight line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$, if $c^2 = a^2(1 + m^2)$ and the point of contact of the tangent $y = mx \pm a\sqrt{1 + m^2}$ is $\left(\pm \frac{ma}{\sqrt{1 + m^2}}, \frac{\mp a}{\sqrt{1 + m^2}}\right)$.

Pole and Polar

(1) **Polar of a point with respect to circle :** If through a point $P(x_1, y_1)$ (with in or without a circle) there be drawn any straight line to meet the given circle at *Q* and *R* the locus of the point of intersection of the tangents at *Q* and *R* is called the polar of *P* and *P* is called the pole of the polar.

(i) The equation of the polar of the point $P(x_1, y_1)$ *w.r.t.* the circle $x^2 + y^2 = a^2$ is given by $xx_1 + yy_1 = a^2$.

(ii) And similarly for general equation of the circle $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.



The pole of line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 = a^2$ is $(-la^2/n, -ma^2/n)$.

(3) **Conjugate points :** Two points *A* and *B* are conjugate points with respect to a given circle if each lies on the polar of the other with respect to the circle.

(4) **Conjugate lines :** If two lines be such that the pole of one lies on the other, then they called conjugate lines with respect to the given circle.

Equation to the chord bisected at a given point

The equation of the chord of circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S'$ *i.e.* $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Equation of common chord of two circles

Equation of the common chord of two circles $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is given by $S_1 - S_2 = 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$.

Angle of intersection of two circles

If θ be the angle at which two circle of radii r_1 and r_2 intersect, then, $\cos\theta = r_1^2 + r_2^2 - d^2 / 2r_1r_2$, where d is distance between their centres.

Condition for orthogonal intersection : $r_1^2 + r_2^2 = d^2$ **or** $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

Equation of normal

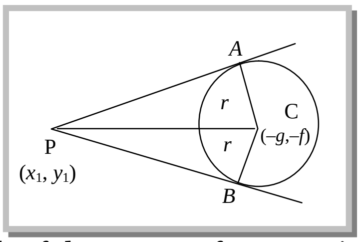
(1) The equation of the normal to the circle $x^2 + y^2 = a^2$ at point (x_1, y_1) is $\frac{x}{x_1} - \frac{y}{y_1} = 0$.

(2) The equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is given by $\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$.

(3) The normal of any point of a circle always passes through the centre of the circle.

Length of the tangent from a point to a circle

The length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.



Note : ❑ If PT is the length of the tangent from a point P to given circle, then PT^2 is called the power of the point with respect to the given circle.

Combined equation of pair of tangents

As that two tangents can be drawn from a point $P(x_1, y_1)$ to a given circle. The combined equation of the pair of tangents drawn from a point (x_1, y_1) to a circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by,

$$(x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = \{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\}^2 \text{ or } SS' = T^2.$$

Chord of contact of tangents

(1) **Chord of contact :** The chord joining the points of contact of the two tangents to a circle drawn from a given point, outside it, is called the chord of contact of tangents.

(2) **Equation of chord of contact :** The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

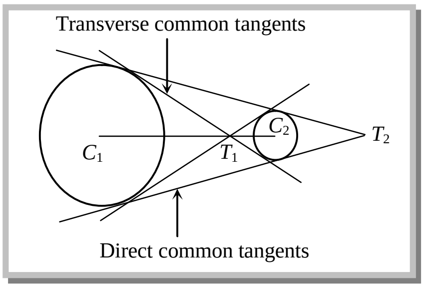
Note : ❑ The equation of the chord of contact of tangents drawn from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

System of circles

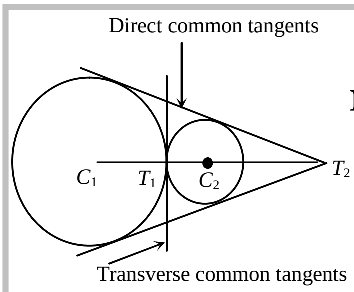
(1) **Common tangents of two circles :** (i) **Different cases of intersection of two circles :** Let the equations of the two circles be $(x - h_1)^2 + (y - k_1)^2 = a_1^2$ .....(i)

and $(x - h_2)^2 + (y - k_2)^2 = a_2^2$ .....(ii)

with the centres $C_1(h_1, k_1)$ and $C_2\ (h_2, k_2)$ and radii a_1 and a_2 respectively. Then the following cases of intersection of these two circles may arise :

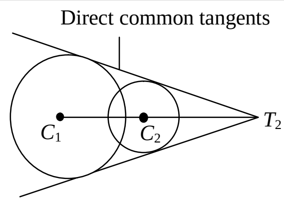


Case I : When $C_1C_2 > a_1 + a_2$ *i.e.* the distance between the centres is greater than the sum of the radii. In this case, the two circles do not intersect with each other and four common tangents can be drawn to two circles. Two of them are direct common tangents and the two are transverse common tangents. The points of intersection of direct and tansverse common tangents always lie on the line joining the centered C_1 and C_2 divide it internally and externally respectively in the ratio $a_1 : a_2$ *i.e.*, $C_1T_2 / C_2T_2 = a_1 / a_2$ (externally) and $C_1T_1 / C_2T_1 = a_1 / a_2$ (internally).



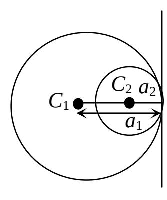
Number of common tangents = 3

Case II : When $C_1C_2 = a_1 + a_2$ *i.e.* the distance between the centres is equal to the sum of the radii. In this case, two direct tangent are real and distinct while the transverse tangents are coincident.



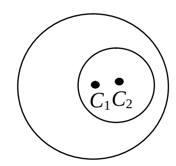
Number of common tangents = 2

Case III : When $C_1C_2 < a_1 + a_2$ *i.e.* the distance between the centres is less than the sum of radii. In this case, the two direct common tangents are real while the transverse tangents are imaginary.



Number of common tangents = 1

Case IV: When $C_1C_2 = a_1 - a_2$ *i.e.* the distance between the centres is equal to the difference of the radii. In this case, two tangents are real and coincident while the other two tangents are imaginary.



Case V : When $C_1C_2 < | a_1 - a_2 |$ *i.e.*, the distance between the centres is less than the difference of the radii. In this case, all the four common tangents are imaginary.

Radical axis and Radical centre

(1) **Radical axis :** The radical axis of two circles is the locus of a point which moves in such a way that the lengths of the tangents drawn from in to the two circles are equal.

Let the two circles be $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ .....(i) and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ .....(ii)

The equation of the radical axis of the two circle is $S_1 - S_2 = 0$ i.e., $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$.

(2) **Radical centre :** The point of intersection of the radical axes of three circle whose centres are non-collinear, taken in pairs, is called the radical centre of the circles.

Coaxial circles

A system of circles; every pair of which has the same radical axis is called a coaxial system of circles. Circles passing through two fixed points form a *co*-axial system of circles, because every pair of circles has the same common chord and therefore the same radical axis.

(1) **The equation of a system of co-axial circles when the equation of the radical axis and of one circle of the system are given:** Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle and $P = lx + my + n = 0$ be the radical axis, then $S + \lambda P = 0$ (λ is an arbitary constant) represents the co-axial system of circles.

(2) **The equation of a co-axial system of circles when the equation of any two circles of the system are given:** Let $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ be two circles of the *co*-axial system. Then $S_1 + \lambda S_2 = 0 (\lambda = -1)$ represents the co-axial system.

(3) **The equation of a system of coaxial circles in the simplest form** : The simplest form of the equation of a *co*-axial system of circles can be put in the form $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant. The common radical axis of this system is y-axis.

Some important points

(1) If two given circles intersect each other, then the radical axis is same as the common tangent.

(2) The pair of tangents from (0,0) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at right angles, if $g^2 + f^2 = 2c$.

(3) The area of triangle formed by the tangent (x_1, y_1) on the circle $x^2 + y^2 = a^2$ with the *co*-ordinate axis is $a^4 / 2|x_1y_1|$.

(4) The angle between the tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\frac{a}{\sqrt{S_1}}$.

(5) Let $S_1 = 0, S_2 = 0$ be two circles with radii r_1, r_2 then $\frac{S_1}{r_2} \pm \frac{S_2}{r_2} = 0$ will meet at right angles.

(6) If x_1, x_2 are the roots of $x^2 + ax + b = 0$ and y_1, y_2 are the roots of $y^2 + cy + d = 0$, then the circle having the line joining (x_1, y_1); (x_2, y_2) as diameter is $(x^2 + ax + b) + (y^2 + cy + d) = 0$.

(7) The area of the triangle formed by the tangents from the points (h, k) to the circle $x^2 + y^2 = a^2$ and their chord of contact is $\frac{a}{h^2 + k^2}(h^2 + k^2 - a^2)^{3/2}$.

(8) If the line $lx + my + n = 0$ is a tangent to the circle $(x - h)^2 + (y - k)^2 = a^2$ then $(hl + km + n)^2 = a^2(l^2 + m^2)$.