PLANE

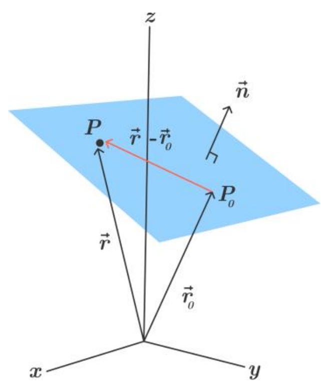
Contents

1.1. PLANE

1.1.1 Introduction

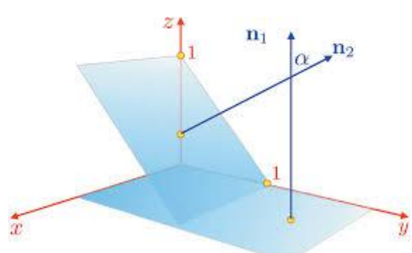
A **plane** is a flat, two-dimensional surface that extends indefinitely. A plane is the two dimensional analogue of a point (zero dimensions), a line (one dimension) and three-dimensional space. Planes can arise as subspaces of some higher-dimensional space, as with one of a room's walls, infinitely extended, or they may enjoy an independent existence in their own right, as in the setting of two-dimensional Euclidean geometry.

When working exclusively in two-dimensional Euclidean space, the definite article is used, so *the* plane refers to the whole space. Many fundamental tasks in mathematics, geometry. Trigonometry, graph theory, and graphing are performed in a two-dimensional space, often in the plane.



1.1.2 Plane

A plane is a surface such that if any two points are taken on it, the straight line joining them lies wholly on the surface, that is all the points on the joining line also lie on the plane. The normal to a plane is a straight line which is perpendicular to any line which lies on the plane.



1.1.2.1 Equation of a Plane

(A) Cartesian form

• **General form**: A Plane is represented by an equation of first degree: $ax+by+cz+d=0$, where a,b,c are not all zero and are the direction ratios of the normal to the plane. Conversely, every equation of the first degree in x, y, z represents a plane.

• **Intercept form**: If a plane makes intercepts a, b, c on the co-ordinate axes, its equation is:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

• **Normal form**: lx + my+ n=p, where l, m, n are the direction cosines of the normal to the plane and p is the length of the perpendicular from the origin to the plane.

Note: (a) Equation of the yz-plane is x=0
(b) Equation of the zx-plane is y=0
(c) Equation of the xy-plane is z=0

(B) Vector form

- A plane which passes through the point with position vector $\vec{a}$ and which is parallel to the vectors $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+t\vec{b}+p\vec{c}$ where, t and p are scalar parameters.
- A plane passing through two given points with position vectors $\vec{a}$ and $\vec{b}$ and parallel to the vector $\vec{c}$ is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})+p\vec{c}$, where t and p are scalar parameters.
- A plane passing through three points with position vectors $\vec{a},\vec{b},\vec{c}$ is:
  $\vec{r}.(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})=[\vec{a}\vec{b}\vec{c}]$
- Normal form of the equation of a plane is: $\vec{r}.\vec{n}=p$, where $\vec{n}$ is the unit vector normal to the plane and p is the length of the perpendicular from the origin to the plane.
- A plane normal to the vector $\vec{n}$ and passing through a point with position vector $\vec{a}$ is: $(\vec{r}-\vec{a}).\vec{n}=0$

1.1.2.2 System of Planes

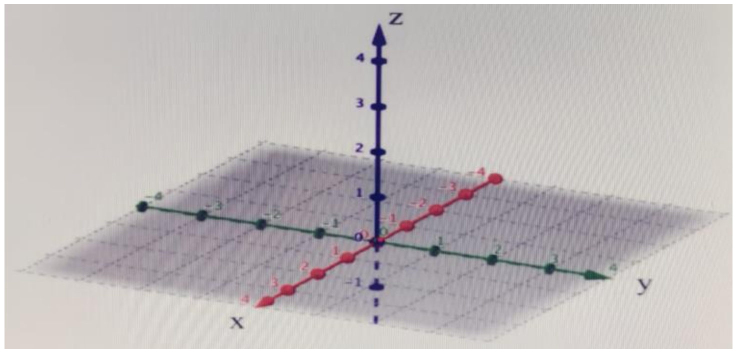
Since equation of a plane contains three independent constants, therefore a plane is uniquely determined by three independent conditions. If we consider a plane satisfying less than three independent conditions, it gives rise to a family of planes all of which satisfy some common condition(s).

(i) Equation of a family of planes parallel to the plane ax+by+cz+d=0 is: ax+by+cz+k=0 where, k is a parameter.

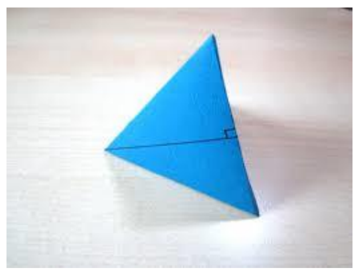
(ii) Equation of a family of planes perpendicular to the line $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ is: ax+by+cz+k=0, where, k is a parameter.

(iii) Equation of a family of planes passing through the intersection of two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is:
$(a_1x+b_1y+c_1z+d_1)+k(a_2x+b_2y+c_2z+d_2)=0$, where, k is a parameter.

(iv) Equation of a family of planes passing through the point (x_1,y_1,z_1), is:
$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ where ratios of A, B, C are the two parameters.



1.1.2.3 Angle between two planes



Angle θ between the planes $a_1x + b_1y + c_1z + d_1 = 0, a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

Vector form: Angle between two planes $\vec{r}.\overrightarrow{n_1} = p_1$ and $\vec{r}.\overrightarrow{n_2} = p_2$ is $\cos^{-1}\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$.

1.1.2.4 Conditions for two planes to be Parallel or Perpendicular

The Planes $a_1x + b_1y + c_1z + d_1 = 0, a_2x + b_2y + c_2z + d_1 = 0$ are:

Parallel If: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$ vector form: $\frac{\overrightarrow{n_1}}{\overrightarrow{n_2}} = \lambda$

Perpendicular if: $a_1a_2 + b_1b_2 + c_1c_2 = 0$ vector form: $\overrightarrow{n_1}.\overrightarrow{n_2} = 0$

1.1.2.5 Position of a point with respect to a plane

Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie with respect to the plane ax+by+cz+d=0 on: same side if: $\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d} > 0$

opposite sides if: $\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d} < 0$

1.1.2.6 Distance of a point from a plane

The perpendicular distance of the point (x_1, y_1, z_1) from the plane:

ax+by+cz+d=0 is: $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{(a^2+b^2+c^2)}}$

lx+my+nz=p is: $| p - lx_1 - my_1 - nz_1 |$

1.1.2.7 Bisectors of the Angles between Two Planes

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be the equations of two planes, written in such a way that d_1 and d_2 are both positive. Then the equations of the planes bisecting the angles between the given planes are:

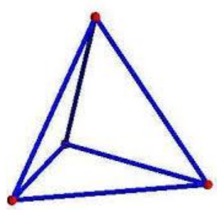
$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

Out of the above two equations of the bisecting planes, the one which:

- Contains the origin: corresponds to + sign
- Does not contain the origin: corresponds to – sign
- Bisects the acute angle between the planes: is the one which makes an angle of less than 45° with both the given planes.

1.1.2.8 Tetrahedron

A tetrahedron (plural: tetrahedron or tetrahedrons), also known as a triangular pyramid, is a polyhedron composed of four triangular faces, six straight edges, and four vertex corners.

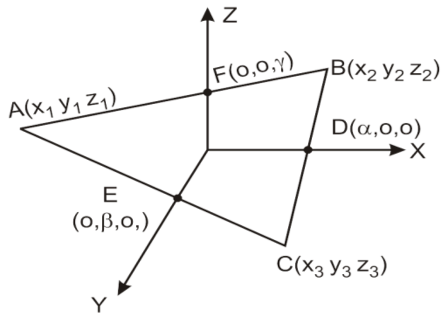


Tetrahedron

A tetrahedron is formed by the intersection of four planes. If $A_i(x_i, y_i, z_i), i = 1,2,3,4$ are the four vertices of a tetrahedron, then:

Volume of tetrahedron = $\frac{1}{6} \left| \begin{vmatrix} x_1 y_1 z_1 1 \\ x_2 y_2 z_2 1 \\ x_3 y_3 z_3 1 \\ x_4 y_4 z_4 1 \end{vmatrix} \right|$

Illustration: A triangle *ABC* is placed so that the midpoints of its sides are on the x, y and z axes respectively. Lengths of the intercepts made by the plane containing the triangle on these axes are respectively α, β, Υ. Find the coordinates of the centroid of the triangle *ABC* .



Solution: Equation of the plane containing the triangle *ABC* is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\Upsilon}=1$ which meets the axes at $(\alpha,0,0),(0,\beta,0)$ and $(0,0,\Upsilon)$.
Let the coordinates of *A, B, C* be $x_{1,}y_{1,}z_1$, $x_{2,}y_{2,}z_2$, $x_{3,}y_{3,}z_3$ respectively.

Mid-point of *AB* = $(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2})=(0,0,\Upsilon)$

Mid-point of *BC* = $(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},\frac{z_2+z_3}{2})=(\alpha,0,0)$

Mid-point of A*C* = $(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2},\frac{z_1+z_3}{2})=(0,\beta,0)$

Solving the above equations, the co-ordinates of vertices are:

A = $(-\alpha,\beta,\Upsilon)$, B = $(\alpha,-\beta,\Upsilon)$, C = $(\alpha,\beta,-\Upsilon)$.

Hence coordinates of the centroid of the triangle *ABC* =$(\alpha/3,\beta/3,\Upsilon/3)$.

Illustration: Find the equation of the plane through the intersection of the planes $x+2\text{y}+3\text{z}-4=0$ and $2\text{x}+y-z+5=0$ and perpendicular to the plane $5\text{x}+3\text{y}+6\text{z}+8=0$.

Solution: Any plane through the intersection of the given planes is $(x+2\text{y}+3\text{z}-4)+\lambda(2\text{x}+y-z+5)=0$ Or $x(1+2\lambda)+y(2+\lambda)+z(3-\lambda)-(4-5\lambda)=0$
Since the plane above is perpendicular to the plane $5\text{x}+3\text{y}+6\text{z}+8=0$
$\therefore 5(1+2\lambda)+3(2+\lambda)+6(3-\lambda)=0$ which gives $\lambda=-29/7$.
Putting the value of λ, we get the equation of the required plane as $51\text{x}+15\text{y}-50\text{z}+173=0$

Illustration: Find equation of bisector planes of angle between the planes x + 2y + 2z = 9 and 4x – 3y +12z +13 = 0. Further specify
(i) which of these is the bisector of obtuse angle.
(ii) which of these bisects the angle that contains the origin.

Solution: Writing in the form d_1, $d_2 > 0$, the equations of the planes are:
-x - 2y - 2z + 9 = 0 and 4x – 3y +12z +13 = 0

The required equations of the bisector planes are given by:

$$\frac{-x-2y-2z+9}{\sqrt{(1+4+4)}} = \pm\frac{4x-3y+12z+13}{\sqrt{(16+9+144)}}$$

Hence the equations of the bisecting planes are:

with + sign : 25x+17y+62z-78 = 0
with – sign : x+35y-10z-156 = 0

Hence the bisecting plane containing the origin is: 25x+17y+62z-78 = 0

Finding Obtuse angle bisector:

Angle between the plane x + 2y + 2z = 9 and one of the bisecting plane x+35y-10z-156= 0 is:

$$\cos\theta = \frac{1\times1+2\times35-2\times10}{\sqrt{(1^2+2^2+2^2)\sqrt{(1^2+35^2+10^2)}}} = \frac{17}{36}$$

$$\tan\theta = \frac{\sqrt{1007}}{17} > 1 \qquad \text{I.e.}\,\theta > 45d$$

Hence the plane which bisects the obtuse angle is: x+35y-10z-156 = 0

In three dimensional space, the location of a point can be uniquely specified using a reference frame of three mutually perpendicular axis X, Y and Z. The location of a point is specified in the form (x,y,z) where x, y and z are the distances of the points along X,Y and Z axis respectively. The three axis meet at a common point called the origin (0,0,0).

The three-axis partition the 3-D space into 8 parts called octants. The sign of x, y and z co-ordinate may be + or – depending upon the particular octant in which the point is located.