Motion in Two Dimension

Two Dimensional Motion

The motion of an object is called two dimensional, if two of the three co-ordinates required to specify the position of the object in space changes *w.r.t.* time. Two special cases of motion in two dimension are

(1) Projectile motion, (2) Circular motion

Projectile

A body which is in flight through the atmosphere but is not being propelled by any fuel is called projectile.

Example: (i) A bomb released from an aeroplane in level flight (ii) A bullet fired from a gun (iii) An arrow released from bow (iv) A Javelin thrown by an athlete.

Component of Projectile Motion

The motion of a projectile is a two-dimensional motion. So, it can be discussed in two parts.

(i) Horizontal motion (ii) Vertical motion

These two motions take place independent of each other. This is called the principle of physical independence of motions. The velocity of the particle can be resolved into two mutually perpendicular components.

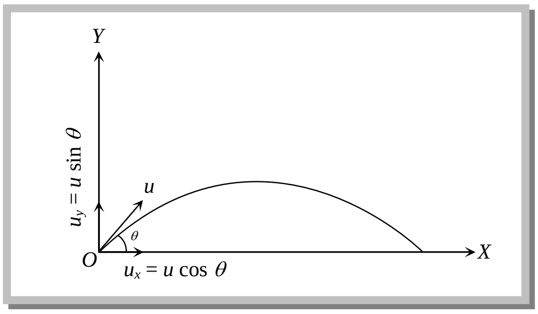
Horizontal component and vertical component.

The horizontal component remains unchanged throughout the flight. The force of gravity continuously affects the vertical component. Thus, while the horizontal motion is a uniform motion, the vertical motion is a uniformly accelerated motion.

Oblique Projectile Motion

(1) **Equation of trajectory :** A projectile thrown with velocity u at an angle θ with horizontal, then it's equation of motion $y = x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2\cos^2\theta}$ or

$$y = x\tan\theta\left[1 - \frac{x}{R}\right]$$



(2) **Instantaneous velocity :** $v = \sqrt{u^2 + g^2t^2 - 2u\,gt\sin\theta}$

(3) **Direction of instantaneous velocity :**

$$\tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$$

(4) **Time of flight :** The total time taken by the projectile to go up and come down to the same level from which it was projected is called time of flight. Now as time taken to go up is equal to the time taken to come down, so

$$\text{Time of flight } T = 2t = \frac{2u\sin\theta}{g} = \frac{2u_y}{g}$$

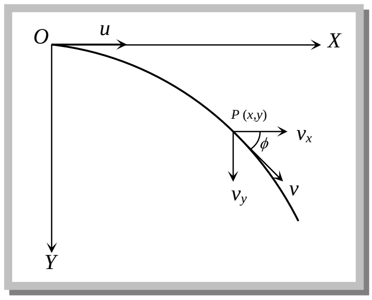
(5) **Horizontal range :** It is the horizontal distance traveled by a body during the time of flight.

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u_x u_y}{g}$$ (where u_x and u_y are the horizontal and vertical component of initial velocity)

(6) **Maximum height :** $H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{u_y^2}{2g}$

Horizontal Projectile Motion

(1) **Equation of horizontal projectile :** $y = \dfrac{1}{2}\dfrac{gx^2}{u^2}$



(2) **Instantaneous velocity :** $v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + g^2t^2}$

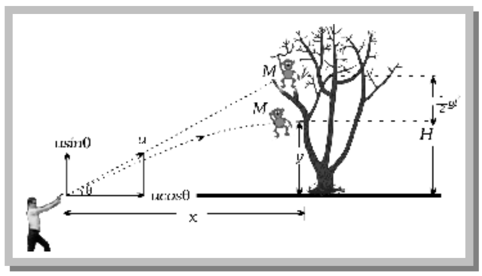
(3) **Direction of instantaneous velocity :** $\tan\phi = \dfrac{v_y}{v_x}$ or $\phi = \tan^{-1}\left(\dfrac{gt}{u}\right)$

(4) **Time of flight :** $T = \sqrt{\dfrac{2h}{g}}$

(5) **Horizontal range :** $R = u\sqrt{\dfrac{2h}{g}}$

Important Points about Projectile Motion

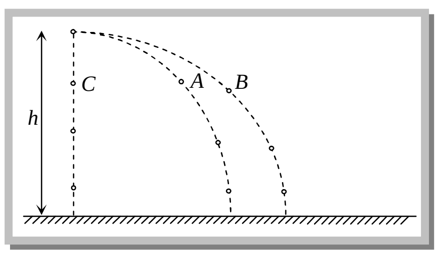
(1) A hunter aims his gun and fires a bullet directly towards a monkey sitting on a distant tree. At the instant the bullet leaves the barrel of the gun, the monkey drops from the tree freely.



The bullet will hit the monkey because the vertical distance travelled by the monkey in downward direction is exactly equal to the amount the bullet falls in the same time.

(2) In projectile motion, horizontal component of velocity ($u\cos\theta$), acceleration (g) and mechanical energy remains constant while, speed, velocity, vertical component of velocity ($u\sin\theta$), momentum, kinetic energy and potential energy all changes. Velocity, and KE are maximum at the point of projection while minimum (but not zero) at highest point.

(3) If projectiles A and B are projected horizontally with different initial velocity from same height and third particle C is dropped from same point then



(i) All three particles will take equal time to reach the ground.

(ii) Their net velocity would be different but all three particle possess same vertical component of velocity.

(iii) The trajectory of projectiles A & B will be straight line *w.r.t.* particle *C*.

(4) **Change in velocity :** (Between projection point and highest point) $= -u\sin\theta$

$$(\text{Between complete projectile motion}) = -2u\sin\theta$$

(5) **Change in momentum :** $(\text{Between projection point and highest point}) = -mu\sin\theta$

$$(\text{Between complete projectile motion}) = -2mu\sin\theta$$

(6) If angle of projection is changed from θ to $\theta' = (90 - \theta)$, the range remains unchanged. These angle θ and $90 - \theta$ are called complementary angles of projection.

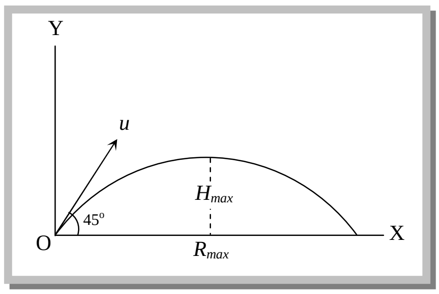
(7) For angle of projection $\theta_1 = (45 - \alpha)$ and $\theta_2 = (45 + \alpha)$; Range will be same and equal to $\frac{u^2\cos 2\alpha}{g}$. Angle θ_1 and θ_2 are also the complementary angles.

(8) For Angle of projection θ and $90 - \theta$ (complementary angles)

Ratio of range $\frac{R_1}{R_2} = 1$; Ratio of time of flight $= \frac{T_1}{T_2} = \tan\theta$; Ratio of Maximum Height $= \frac{H_1}{H_2} = \tan^2\theta$

Multiplication of time of flight $= T_1T_2 = \frac{2u\sin\theta}{g}\frac{2u\cos\theta}{g} = \frac{2R}{g}$

(9) A projectile will have maximum range when it is projected at an angle of 45° to the horizontal and the maximum range will be (u^2/g).



When the range is maximum, the height H reached by the projectile

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin^2 45}{2g} = \frac{u^2}{4g} = \frac{R_{\max}}{4}$$

(10) For Height H, To be maximum, $\sin^2\theta = (\max) = 1$ *i.e.* $\theta = 90°$

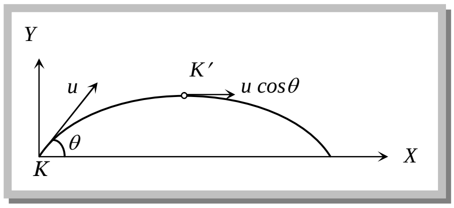
So, $$H_{\max} = \frac{u^2}{2g}$$

(11) When a projectile moves upward its kinetic energy decreases, potential energy increases but the total energy always remain constant. At the highest point of trajectory

$$KE = \frac{1}{2}mv^2\cos^2\theta,\ PE = \frac{1}{2}mu^2\sin^2\theta.$$

Total energy $= KE + PE = \frac{1}{2}mu^2\cos^2\theta + \frac{1}{2}mu^2\sin^2\theta = \frac{1}{2}mu^2$ = Energy at the point of projection.

(12) If a body is projected with initial kinetic energy K, with angle of projection θ with the horizontal then at the highest point of trajectory its kinetic energy will be $K' = K\cos^2\theta$



(13) If in case of projectile motion range R is n times the maximum height H. Then $\theta = \tan^{-1}[4/n]$

(14) If a projectile of mass m thrown with velocity u making angle θ with horizontal from point O and returns to the ground at G. H is the highest point attained by it, then

(i) **In going from O to H, following change take place**

(a) Change in velocity $= u\sin\theta$

(b) Change in speed $= u(1-\cos\theta) = 2u\cos^2\theta/2$

(c) Change in momentum $= m\,u\sin\theta$

(d) Change (loss) in kinetic energy $= 1/2\; mu^2\sin^2\theta$

(e) Change (gain) in potential energy $= 1/2\; mu^2\sin^2\theta$

(f) Change in the direction of motion $= \angle\theta$

(ii) **On return to the ground, that is in going from O to G, the following changes take place**

(a) Change in speed = zero

(b) Change in velocity = $2u\sin\theta$

(c) Change in momentum = $2\,mu\sin\theta$

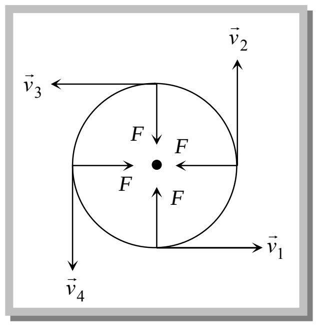
(d) Change in kinetic energy = zero

(e) Change in potential energy = zero

(f) Change in the direction of motion $= \angle 2\theta$

Circular Motion

Circular motion is another example of motion in two dimensions. To create circular motion in a body it must be given some initial velocity and a force must then act on the body always directed at right angles to instantaneous velocity.



Since this force is always at right angles to the displacement due to the initial velocity therefore no work is done by the force on the particle. Hence, its kinetic energy and thus speed is unaffected. But due to simultaneous action of the force and the velocity the particle follows resultant path, which in this case is a circle. Circular motion can be classified into two types – Uniform Circular Motion and Non-uniform Circular Motion

(1) **Uniform circular motion :** When a point object is moving on a circular path with a constant speed (*i.e.* it covers equal distances on the circumference of the circle in equal intervals of time), then the motion of the object is said to be a uniform circular motion. In uniform circular motion, the velocity of the object (represented by the tangent to the circular path at a given instant) is changing its direction continuously, hence it is a case of uniformly accelerated motion.

Angular displacement (θ) : The angles turned by a body moving on a circle from some reference line is called angular displacement.

Unit = Radian or degree

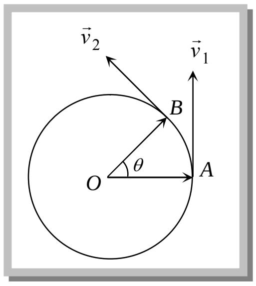
Dimension = $M^0L^0T^0$

$2\pi\,rad = 360^o$ = 1 revolution

Angular displacement is a axial vector quantity.

Relation between linear displacement and angular displacement : $s = r\theta$ or $\vec{s} = \vec{\theta} \times \vec{r}$

Angular velocity (ω) : Angular velocity of an object in circular motion is defined as the time rate of change of its angular displacement. $\omega = \frac{d\theta}{dt}$



Unit = $rad\text{-}s^{-1}$

Dimension = $M^0L^0T^{-1}$

Angular velocity is a axial vector.

Change in velocity : $\Delta \vec{v} = \vec{v}_2 - \vec{v}_1$

$$\Rightarrow \left|\Delta \vec{v}\right| = \left|\vec{v}_2 - \vec{v}_1\right| = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\theta} = \sqrt{2v^2(1-\cos\theta)} = 2v\sin\frac{\theta}{2} \text{ (As } v_1 = v_2 = v\text{)}$$

Relation between linear velocity and Angular velocity $\vec{v} = \vec{\omega} \times \vec{r}$ or $v = \omega r$

Time period (T) : In circular motion, the time period is defined as the time taken by the object to complete one revolution on its circular path.

Unit = second

Dimension = $M^0L^0T^1$

Frequency (n): In circular motion, the frequency is defined as the number of revolutions completed by the object on its circular path in a unit time.

Unit = s^{-1} or hertz (Hz)

Dimension = $M^0L^0T^{-1}$

Relation between time period and frequency : $T = 1/\upsilon$

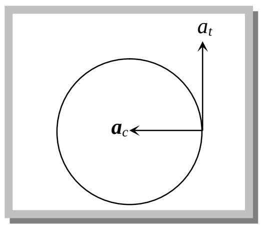
Relation between angular velocity, frequency and time period : $\omega = \frac{2\pi}{T} = 2\pi n$

Angular acceleration : Angular acceleration of an object in circular motion is defined as the time rate of change of its angular velocity. $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

(i) Unit = $rad.\ s^{-2}$ (ii) Dimension $= M^0L^0T^{-2}$

Relation between linear acceleration and angular acceleration : $\vec{a} = \vec{\alpha} \times \vec{r}$ or $a = r\alpha$

Centripetal acceleration : Acceleration acting on the object undergoing uniform circular motion is called centripetal acceleration. It always acts on the object along the radius towards the centre of the circular path.



$$a = \omega^2 r = 4\pi^2 n^2 r = \frac{4\pi^2}{T^2} r = \frac{v^2}{r}$$

Tangential and centripetal acceleration : $\vec{a} = \vec{a}_t + \vec{a}_c = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\upsilon}$

Thus the resultant acceleration of the particle at P has two component accelerations where $|\vec{a}_t| = \alpha r.$ and $|\vec{a}_c| = \omega \upsilon = \omega(\omega r) = \omega^2 r = \upsilon^2 / r$

Equations of circular motion :

(i) $\omega_2 = \omega_1 \pm \alpha t$

(ii) $\theta = \omega_1 t \pm \frac{1}{2}\alpha t^2$

(iii) $\omega_2^2 = \omega_1^2 \pm 2\alpha t$

(4) $\theta_n = \omega_1 \pm \frac{\alpha}{2}(2n-1)$

where ω_1 = Initial angular velocity of particle,

ω_2 = Final angular velocity of particle

α = Angular acceleration/retardation of particle

θ = Angle covered by the particle in time t

θ_n = Angle covered by the particle in n^{th} second

Centripetal force :

(i) **Formula :** $F = \frac{mv^2}{r} = m\omega^2 r = 4\pi^2 mn^2 r = \frac{4\pi^2 mr}{T^2}$

(ii) **Centripetal force in different situation**

	Situation	Centripetal force
(a)	Particle tied to string, moving in a horizontal circle.	Tension in the string.
(b)	Vehicle taking a turn on a level road.	Frictional force exerted by the road on the tyres.
(c)	A vehicle on a speed breaker.	Weight of the body or a component of weight.
(d)	Revolution of earth around the sun.	Gravitational force exerted by the sun.
(e)	Electron revolving around the nucleus in an atom.	Coulomb attraction exerted by the protons in the nucleus.
(f)	A charged particle describing a circular path in a magnetic field.	Magnetic force exerted by the agent that sets up the magnetic field.

Centrifugal force : It is an imaginary force to incorporated effects of inertia. Centrifugal force is not a force of reaction and it comes into play only when the necessary centripetal force is absent. The centrifugal force is equal and opposite to centripetal force and both forces act on same body. Still centripetal and centrifugal forces do not neutralize each other because they act at difference instants of time on the same body. The centrifugal force comes into play only when the necessary centripetal force is not being fully provided.

Important points about uniform circular motion :

(i) **Work done by centripetal force :** It is always zero as it is perpendicular to velocity and hence instantaneous displacement.

Example : When a satellite established once in a orbit around the earth and it starts revolving with particular speed, then no fuel is required for its circular motion.

(ii) **Skidding of vehicle :** When a vehicle turns on a circular path it requires centripetal force. If friction provides this centripetal force then vehicle can move in circular path safely if

Friction force ≥ Required centripetal force $\Rightarrow \mu mg \geq \frac{mv^2}{r} \Rightarrow v_{safe} \leq \sqrt{\mu rg}$

This is the maximum speed by which vehicle can turn in a circular path of radius r, where coefficient of friction between the road and tyre is μ.

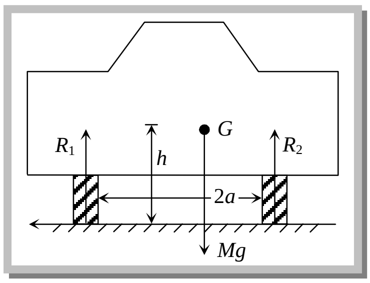
(iii) **Skidding on rotating platform :** To avoid the skidding of an object placed at a distance r from axis of rotation, the maximum angular velocity of the platform, $\omega = \sqrt{(\mu g/r)}$, where μ is the coefficient of friction between the object and the platform.

(iv) **Bending of a cyclist :** A cyclist provides himself the necessary centripetal force by leaning inward on a horizontal track, while going round a curve. Consider a cyclist of weight mg taking a turn of radius r with velocity v. In order to provide the necessary centripetal force, the cyclist leans through angle θ.

$$\tan\theta = \frac{v^2}{rg}$$

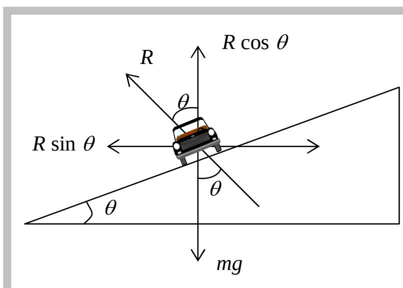
(v) **Overturning of vehicle :** The max speed of a car without overturning on a flat road is given by $v = \sqrt{\frac{gra}{h}}$

where $2a$ = Distance between the centre of wheels of the car, h = Height of the centre of gravity of the car from the road level.



If a body is moving on a curved road with speed greater than the speed limit, the reaction at the inner wheel disappears and it will leaves the ground first.

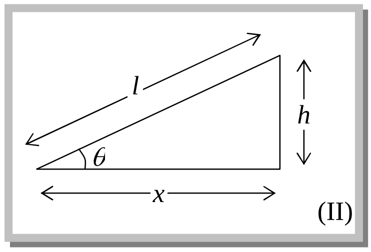
(vi) **Banking of a road :** For getting a centripetal force cyclist bend towards the centre of circular path but it is not possible in case of four



(I)

wheelers, therefore outer bed of the road is raised so that a vehicle moving on it gets automatically inclined towards the centre. $\tan\theta = \frac{v^2}{rg}$

or $\quad \tan\theta = \frac{\omega^2 r}{g} = \frac{v\omega}{rg} = \frac{h}{l}$ [since θ is very small]

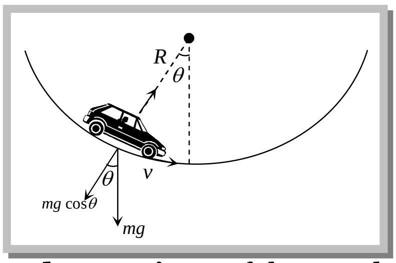


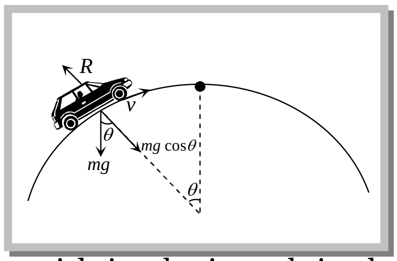
If friction is also present between the tyres and road then

$$\frac{v^2}{rg} = \frac{\mu + \tan\theta}{1 - \mu\tan\theta}$$

Maximum safe speed on a banked frictional road $v = \sqrt{\frac{rg(\mu + \tan\theta)}{1 - \mu\tan\theta}}$

Reaction by the road : When car moves on a concave bridge then $R = mg\cos\theta + \frac{mv^2}{r}$ and When car moves on a convex bridge $R = mg\cos\theta - \frac{mv^2}{r}$





(2) **Non-uniform circular motion :** If the speed of the particle in a horizontal circular motion changes with respect to time, then its motion is said to be non-uniform circular motion. The particle simultaneously possesses two accelerations – radial (centripetal) acceleration $\vec{a}_c$ and tangential acceleration $\vec{a}_t$. The resultant acceleration of the particle is given by $\vec{a} = \vec{a}_c + \vec{a}_r \Rightarrow |a| = \sqrt{a_c^2 + a_t^2}$

In non-uniform circular motion

(i) $v \neq$ constant, $\omega \neq$ constant

(ii) Work done by centripetal force will be zero since $\vec{F}_c \perp \vec{v}$

(iii) Work done by tangential of force will not be zero since $F_t \neq 0$

(iv) Rate of work done by net force = rate of work done by tangential force *i.e.* $P = \frac{dW}{dt} = \vec{F}_t . \vec{v}$

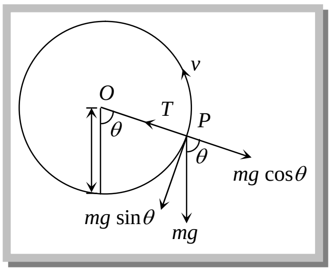
Motion in Vertical Circle

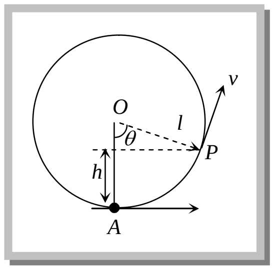
This is an example of non-uniform circular motion. In this motion body is under the influence of gravity of earth. When body move from lowest point to highest point. Its speed decrease and minimum at highest point. Total mechanical energy of the body remains conserved and *KE* converts into *PE* and vice versa.

Velocity at any point on vertical loop : If u is the initial velocity imparted to body at lowest point them. Velocity of body at height h is given by $v = \sqrt{u^2 - 2gl(1 - \cos\theta)}$

Tension at any point on vertical loop : $T = \frac{m}{l}[u^2 - gl(2 - 3\cos\theta)]$

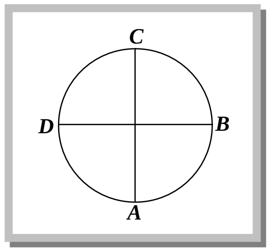
Critical condition for vertical looping : If the tension at *C* is zero, then body will just complete revolution in the vertical circle. This state of body is known as critical state. The speed of body in critical state is called as critical speed.





Various quantities corresponding to critical condition at different points are given in table

Quantity	Point A	Point B	Point C	Point D
Linear velocity (v)	$\sqrt{5gl}$	$\sqrt{3gl}$	$\sqrt{gl}$	$\sqrt{3gl}$
Angular velocity ($\omega = \frac{v}{l}$)	$\sqrt{\frac{5g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}}$	$\sqrt{\frac{3g}{l}}$
Tension in String (T)	6 mg	3 mg	0	3 mg
Kinetic Energy ($KE = \frac{1}{2}mv^2$)	$\frac{5}{2}mgl$	$\frac{3}{2}mgl$	$\frac{1}{2}mgl$	$\frac{3}{2}mgl$
Potential Energy (PE)	0	mgr	2 mgr	mgr
Total Energy $TE = KE + PE$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$



Definition of Vectors

If a physical quantity in addition to magnitude has a specified direction obeys the law of parallelogram of addition, $R = (A^2 + B^2 + 2AB\cos\theta)^{1/2}$ and its addition is commutative, $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ then and then only it is said to be a vector. If any of the above conditions is not satisfied the physical quantity cannot be a vector.

Example : Velocity, force, weight, momentum, impulse, torque, temperature gradient, gravitational field etc.

Note : ❑ If a physical quantity is a vector, it must have a direction but converse may or may not be true. It means if a physical quantity has direction it may or may not be vector.

Example : Pressure, surface tension, current, time *etc*. have direction but are not vectors.

❑ In physics certain physical quantities such as electric dipole moment, current density and area (*i.e.* surface elementary area) are defined as vectors with definite direction.

(1) **Polar-vectors :** These have starting point or point of application.

Example : Displacement, force *etc*.

(2) **Axial-vectors (Rotational vectors) :** These represent rotational effects and are always along the axis of rotation in accordance with right hand screw rule.

Example : Angular displacement, angular velocity, angular acceleration torque and angular momentum, *etc*.

Note : ❑ Null vector or zero vector is defined as a vector whose magnitude is zero and direction indeterminate. It differs from ordinary zero in the sense that ordinary zero has no direction but it has a direction which becomes indeterminate by virtue of its zero magnitude.

❑ Any vector can be written as the product of *unit* vector in that direction and magnitude of the given vector *i.e.* $\vec{A} = A\hat{A}$ or $\hat{A} = \vec{A}/A$.

- Unit vectors along *x*, *y* and *z*- axes are usually represented by $\hat{i}.\hat{j}$ and $\hat{k}$ respectively. So if the co-ordinates of a point are (*x*, y, z), the position vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ with $r^2 = x^2 + y^2 + z^2$
- Angle between collinear vectors is always zero (same direction) or 180° (in opposite direction).

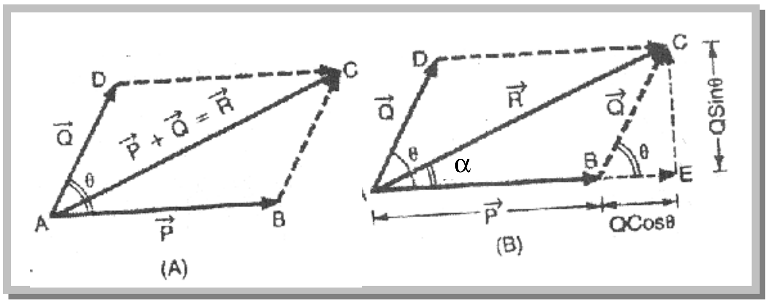
Tensor

Certain physical quantities neither behaves as scalar nor vector, these are called tensors

Example : Stress, moment of inertia, surface tension.

Addition of Vectors

The addition of two vectors can be understood by the law of parallelogram. According to this law if two vectors $\vec{P}$ and $\vec{Q}$ are represented by two adjacent sides of a parallelogram both pointing outwards as shown in Fig. (A), then the diagonal drawn through the intersection of the two vectors represents the resultant (*i.e.* vector sum of $\vec{P}$ and $\vec{Q}$). Note here that if $\vec{Q}$ is displaced from position *AD* to *BC* by displacing it parallel to itself, this method becomes equivalent to the triangle method.



In case of addition of two vectors by parallelogram method in the light of Fig (B), the magnitude of resultant will be given by,

$$(AC)^2 = (AE)^2 + (EC)^2 \quad \text{or} \quad R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2 \Rightarrow R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

and the direction of resultant from $\vec{P}$ will be given by $\tan\alpha = \frac{CE}{AE} = \frac{Q\sin\theta}{P + Q\cos\theta} \Rightarrow \alpha = \tan^{-1}\left[\frac{Q\sin\theta}{P + Q\cos\theta}\right]$

Important Points About Vector Addition

(1) To a vector only a vector of same type can be added and the resultant is a vector of the same type. For example, to a force only a force and not velocity can be added and the resultant will be a force and not any other physical quantity.

(2) Vector addition is commutative, *i.e.* $\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$

(3) Vector addition is associative, *i.e.* $\vec{P} + (\vec{Q} + \vec{R}) = (\vec{P} + \vec{Q}) + \vec{R}$

(4) Vector addition is distributive *i.e.* $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$ or $(m+n)\vec{A} = m\vec{A} + n\vec{A}$

(5) As $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$ so *R* will be maximum when, cos θ(max) = 1, *i.e.* θ = 0°, *i.e.* vectors are like or parallel and $R_{max} = P + Q$.

(6) The resultant will be minimum if, cosθ(min) = – 1, *i.e.* $\theta = 180^o$, *i.e.* vectors are non-parallel and $R_{min} = P - Q$

(7) If the vectors *P* and *Q* are orthogonal, *i.e.* θ = 90° *i.e.* cos 90° = 0

$R = \sqrt{P^2 + Q^2}$ and direction of *R* is given by $\tan\alpha = \frac{Q}{P}$

(8) The resultant of two vectors can have any value from $(P - Q)$ to $(P + Q)$ depending on the angle between them and the magnitude of resultant decreases as θ increases from 0^o to 180^o

(9) As $R_{min} = P - Q$ so if $P \neq Q$ $R_{min} \neq 0$ *i.e.* resultant of two vectors of unequal magnitude can never be zero so minimum number of unequal vectors whose sum can be zero is three.

(10) Let $\vec{P} + \vec{Q} + \vec{R} = 0$ i.e. $\vec{R} = -(\vec{P} + \vec{Q})$

This in turn implies that in case of three vectors the resultant may be zero and it will be only when one vector is equal to the negative of the sum of the remaining two vectors, *i.e.* vectors are coplanar.

From the above it is also clear that the resultant of 3 non-coplanar vectors can never be zero or minimum number of non coplanar vectors whose sum can be zero is four.

(11) Subtraction of a vector from a vector is the addition of negative vector, i.e. $\vec{P} - \vec{Q} = \vec{P} + (-\vec{Q})$.

In case of subtraction of a vector from a vector

(i) $R = [(P)^2 + (Q)^2 + 2PQ\cos(180 - \theta)]^{1/2}$

$R = \sqrt{P^2 + Q^2 - 2PQ\cos\theta}$ [as $\cos(180 - \theta) = -\cos\theta$]

(ii) Subtraction is not commutative, *i.e.* $\vec{P} - \vec{Q} \neq \vec{Q} - \vec{P}$ [but $= -(\vec{Q} - \vec{P})$]

(iii) Change in a vector physical quantity means subtraction of initial vector from the final vector.

Components of a Vector

(1) A vector can have any number of component vectors.

(2) When a vector is splitted into two or three component vectors at right angles to each other the component vectors are called rectangular components of a vector.

(3) If $\vec{A}_x$ and $\vec{A}_y$ are the rectangular components of $\vec{A}$ along *x*-axis and *y* - axis in a plane, then $\vec{A} = A_x\hat{i} + A_y\hat{j}$, where $A = \sqrt{A_x^2 + A_y^2}$ Here $A_x = A\cos\theta$ and $A_y = A\sin\theta$ where θ is the angle which the $\vec{A}$ makes with *x*-axis.

(4) If $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ are the rectangular components of $\vec{A}$ along *x*-axis, *y*-axis and *z*-axis respectively in a given space, then $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

(5) If $\vec{A}$ makes an angles α, β and γ with *x*, y and *z*-axis, then $\cos\alpha = \hat{A}.\hat{i}$; $\cos\beta = \hat{A}.\hat{j}$ and $\cos\gamma = \hat{A}.\hat{k}$; which are called direction cosines of $\vec{A}$. Here, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Multiplication or Division of a Vector by a Scalar

(1) When a vector $\vec{A}$ is multiplied by a scalar S, it becomes a vector $S\vec{A}$.

(2) When a vector $\vec{A}$ is divided by a scalar S, it becomes a vector $\vec{A}/S$.

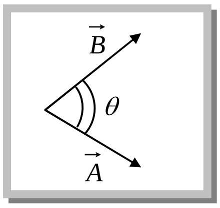
(3) It is worthy to note here that in vector algebra there is no law of division (of a scalar or vector) by a vector, *i.e.* $(S/\vec{V})$ or $(\vec{A}/\vec{V})$ are not defined, so a scalar or a vector, can never be divided by a vector.

Scalar Product of Two Vectors

(1) **Definition :** The scalar product (or dot product) of two vectors is defined as *the product of the magnitude of two vectors with cosine of angle between them.* Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having angle θ between them, then their scalar product written as $\vec{A}.\vec{B}$ is defined $\vec{A}\cdot\vec{B} = AB\cos\theta$

(2) **Properties :**

(i) It is always a scalar which is positive if angle between the vectors is acute (*i.e.*, < 90°) and negative if angle between them is obtuse (*i.e.* 90° < θ < 180°).



(ii) It is commutative, *i.e.* $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$

(iii) It is distributive, *i.e.* $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

(iv) As by definition $\vec{A}\cdot\vec{B} = AB \cos\theta$. So the angle between the vectors $\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{AB}\right]$

(v) Scalar product of two vectors will be maximum when $\cos\theta$ (max) = 1, *i.e.*, $\theta = 0°$, *i.e.* vectors are parallel, and $(\vec{A}\cdot\vec{B})_{max} = AB$.

(vi) Scalar product of two vectors will be minimum when $|\cos\theta|$ (min) = 0, *i.e.*, $\theta = 90°$ $(\vec{A}\cdot\vec{B})_{min} = 0$. *i.e.* if the scalar product of two non-zero vectors vanishes the vectors are orthogonal.

(vii) The scalar product of a vector by itself is termed as self dot product and is given by $(\vec{A})^2 = \vec{A}\cdot\vec{A} = AA\cos\theta = A^2 \Rightarrow A = \sqrt{\vec{A}\cdot\vec{A}}$.

(viii) In case of unit vector $\hat{n} \Rightarrow \hat{n}\cdot\hat{n} = 1\times 1\times\cos 0 = 1$

(ix) In case of orthogonal unit vectors $\hat{i},\hat{j}$ and $\hat{k}$. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ & $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 1\times 1\cos 90 = 0$

(x) In terms of components $\vec{A}\cdot\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)\cdot(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z)$

Which in the light of points (c), (h) and (i) becomes $\vec{A}\cdot\vec{B} = [A_xB_x + A_yB_y + A_zB_z]$.

(3) **Examples of dot product :**

(i) Work : $W = \vec{F}.\vec{S}$, where $\vec{F}$ = Force vector, $\vec{S}$ = Displacement

(ii) Power : $P = \vec{F}.\vec{v}$, where $\vec{v}$ = Velocity

(iii) Magnetic flux : $d\phi = \vec{B}.d\vec{s} \Rightarrow \phi = \int\vec{B}.d\vec{s}$, where $\vec{B}$ = Magnetic induction, $\overrightarrow{ds}$ = Area vector

(iv) Electric flux : $d\phi = \vec{E}.d\vec{s} \Rightarrow \phi = \int\vec{E}\cdot d\vec{s}$, where $\vec{E}$ = Electric intensity

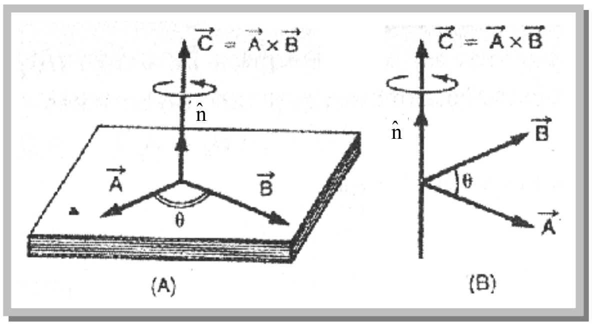
(v) Current : $I = \int\vec{J}.\overrightarrow{ds}$, where $\vec{J}$ = Current density vector

(vi) Potential energy of electric dipole : $U_E = -\vec{P}\cdot\vec{E}$; where $\vec{P}$ = electric dipole moment.

(vii) Potential energy of a magnetic dipole $U_B = -\vec{M}\cdot\vec{B}$.

Vector Product of Two Vectors

(1) **Definition :** The vector product or cross product of two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them, and direction perpendicular to the plane containing the two vectors in accordance with right hand screw rule.

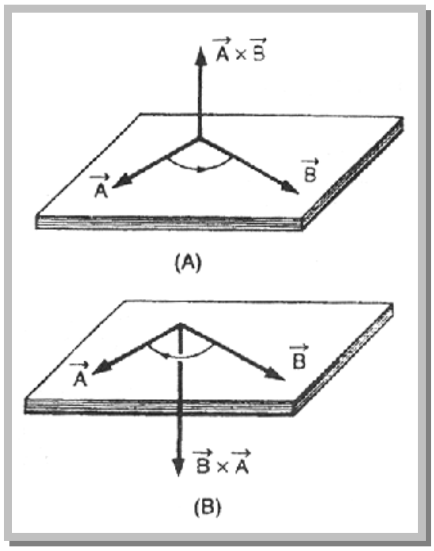


$$\vec{C} = \vec{A} \times \vec{B}$$

Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product written as $\vec{A} \times \vec{B}$ is a vector $\vec{C}$ defined by $\vec{C} = \vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$

(2) **Right handed screw rule :** According to this rule if a right handed screw whose axis is perpendicular to the plane framed by $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A} \times \vec{B}$ *i.e.* $\vec{C}$.

(3) **Properties :** Vector product of two vectors is always a *vector* perpendicular to the plane containing the two vectors, *i.e.,* orthogonal to both the vectors $\vec{A}$ and $\vec{B}$.



(i) Vector product of two vectors is not *commutative, i.e.,* $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ [but $\vec{A} \times \vec{B} = - \vec{B} \times \vec{A}$]

Here it is worthy to note that $|\vec{A} \times \vec{B}| = |\vec{B} \times \vec{A}| = AB \sin\theta$ *i.e.*, in case of vectors $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ magnitudes are equal but directions opposite [See fig.]

(ii) The vector product is *distributive* when the order of the vectors is strictly maintained, *i.e.* $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

(iii) As by definition of vector product of two vectors $\vec{A} \times \vec{B} = AB \sin\theta\ \hat{n}$ so $|\vec{A} \times \vec{B}| = AB \sin\theta$ *i.e.,* $\theta = \sin^{-1}\left[\dfrac{|\vec{A} \times \vec{B}|}{|\vec{A}||\vec{B}|}\right]$

(iv) The vector product of two vectors will be maximum when $\sin\theta$(max) = 1, *i.e.,* $\theta = 90^o$ $|\vec{A} \times \vec{B}|_{max} = AB\hat{n}$ *i.e.,* vector product is maximum if the vectors are orthogonal.

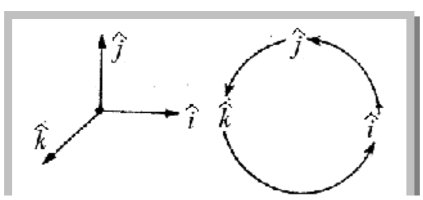
(v) The vector product of two non-zero vectors will be minimum when $|\sin\theta|$ (min) = 0, *i.e.* $\theta = 0^o$ or 180^0 and $[\vec{A} \times \vec{B}]_{min} = 0$ *i.e.* if the vector product of two non-zero vectors vanishes, the vectors are collinear.

(vi) The self cross product, *i.e.*, product of a vector by itself vanishes, *i.e.*, is a null vector, $\vec{A} \times \vec{A} = AA \sin 0\ \hat{n} = \vec{0}$.

(vii) In case of unit vector $\hat{n}$ from point (vi) $\hat{n} \times \hat{n} = 0$ so that $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(viii) In case of orthogonal unit vectors $\hat{i}, \hat{j}, \hat{k}$ in accordance with right hand screw rule.

$$\hat{i} \times \hat{j} = \hat{k}, \qquad \hat{j} \times \hat{k} = \hat{i} \qquad \text{and} \qquad \hat{k} \times \hat{i} = \hat{j}$$



and as cross product is not commutative,

$$\hat{j}\times\hat{i} = -\hat{k}, \quad \hat{k}\times\hat{j} = -\hat{i} \quad \text{and} \quad \hat{i}\times\hat{k} = -\hat{j}$$

(ix) In terms of components $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

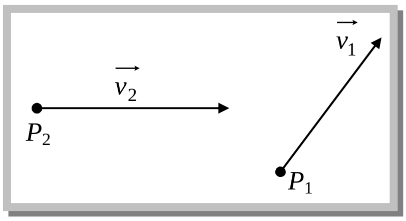
meaning there by $\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$

(x) Examples of cross product : (a) Torque $\vec{\tau} = \vec{r}\times\vec{F}$ (b) Angular momentum $\vec{L} = \vec{r}\times\vec{p}$ (c) Velocity $\vec{v} = \vec{\omega}\times\vec{r}$ (d) $\vec{F} = q(\vec{v}\times\vec{B})$ (e) Torque on a dipole in a field $\vec{\tau}_E = \vec{p}\times\vec{E}$ and $\vec{\tau}_B = \vec{M}\times\vec{B}$.

Note : ❑ $\frac{1}{2}|\vec{A}\times\vec{B}|$ represents the area of a triangle that is made by $\vec{A}$ and $\vec{B}$.

Relative Velocity

The relative velocity of a particle P_1 moving with velocity $\vec{v}_1$ with respect to another particle P_2 moving with velocity $\vec{v}_2$ is given by, $\vec{v}_{r_{12}} = \vec{v}_1 - \vec{v}_2$



(1) If both the particles are moving in the same direction then : $\upsilon_{r_{12}} = \vec{\upsilon}_1 - \vec{\upsilon}_2$

(2) If the two particles are moving in the opposite direction, then : $\upsilon_{r_{12}} = \vec{\upsilon}_1 + \vec{\upsilon}_2$.

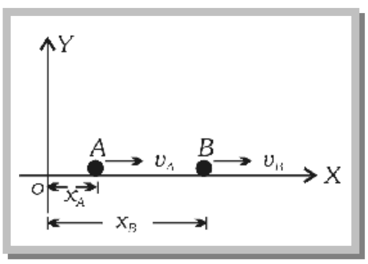
(3) If the two particles are moving in the mutually perpendicular directions, then: $\upsilon_{r_{12}} = \sqrt{\upsilon_1^2 + \upsilon_2^2}$

(4) If the angle between $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ be θ, then $\upsilon_{r_{12}} = \left[\upsilon_1^2 + \upsilon_2^2 - 2\upsilon_1\upsilon_2\cos\theta\right]^{1/2}$

Relative Motion in One Dimension

Let two particles A and B are moving with velocities V_A and V_B and accelerations a_A and a_B respectively. If x_A and x_B are their respective displacements *w.r.t.* to the fixed origin then

Relative displacements of B *w.r.t.* to A, $x_{BA} = x_B - x_A$.



The relative velocity, $v_{BA} = v_B - v_A$

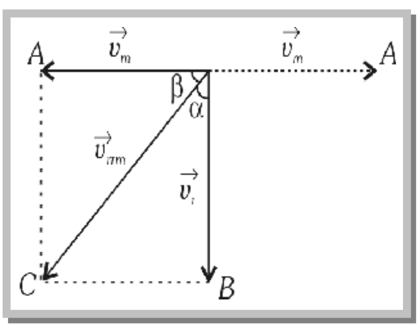
The relative acceleration, $a_{BA} = a_B - a_A$

The equation of motion in one dimension are as $v_{rel} = u_{rel} + a_{rel}t$,

$x_{rel} = u_{rel} + \frac{1}{2}a_{rel}t^2$, $v_{rel}^2 - u_{rel}^2 = 2a_{rel}x_{rel}$.

Relative Velocity of Rain

Suppose the vertically downward velocity of the rain is $\vec{\upsilon}_r$ and a man is walking horizontally with velocity $\vec{\upsilon}_m$.



Relative velocity of the rain *w.r.t.* the man is given by: $\vec{\upsilon}_{rrm} = \vec{\upsilon}_r - \vec{\upsilon}_m$

The parallelogram of vectors shown in the figure.

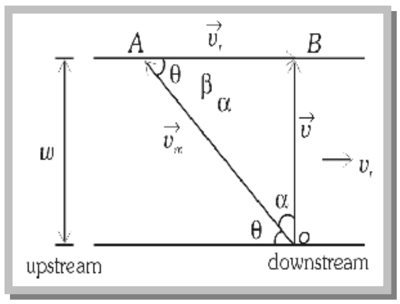
The $\vec{v}_{rrm}$ is given by *OC*. Angle made by $\vec{v}_{rrm}$ with horizontal is given by $\tan\beta = \frac{v_r}{v_m}$ and the angle made by $\vec{v}_{rrm}$ with vertical is given by $\tan\alpha = \frac{v_m}{v_r}$

Note : ❑ To prevent from wetting a man should held the umbrella the direction of v_{rm} where from rain appears to be falling.

Crossing the River

Suppose, the river is flowing with velocity υ_r . A man can swim in still water with velocity υ_m . He is standing on one bank of the river and wants to cross the river two cases arise.

(1) **To cross the river over shortest distance :** That is to cross the river straight, the man should swim making angle θ with the upstream as shown.



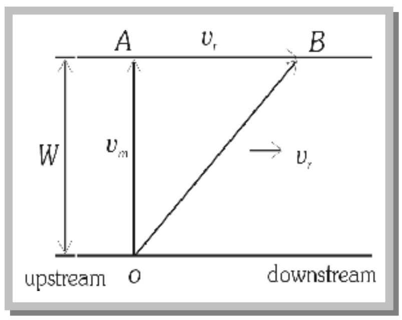
In the fig. here OAB is the triangle of vectors, in which $\vec{OA} = \vec{v}_m, \vec{AB} = \vec{\upsilon}_r$. Their resultant is given by $\vec{OB} = \vec{\upsilon}$. The direction of swimming makes angle θ with upstream. From the triangle *OBA*, we find,

$$\cos\theta = \frac{\upsilon_r}{\upsilon_m} \quad \text{also} \quad \sin\alpha = \frac{\upsilon_r}{\upsilon_m}$$

where α is the angle made by the direction of swimming with the shortest distance (AB) across the river.

Time taken to cross the river **:** If *W* be the width of the river, then time taken to cross the river will be given by: $t_1 = \frac{W}{\upsilon} = \frac{W}{\sqrt{\upsilon_m^2 - \upsilon_r^2}}$

(2) **To cross the river in shortest possible time** – The man should swim perpendicular to the bank.



The time taken to cross the river will be: $t_2 = \frac{W}{\upsilon_m}$

In this case, the man will touch the opposite bank at a distance *AB* down stream. This distance will be given by:

$$AB = \upsilon_r t_2 = \upsilon_r \frac{W}{\upsilon_m} \quad \text{or} \quad AB = \frac{\upsilon_r}{\upsilon_m} W$$

Note : ❑ Velocity of swimmer is always *w.r.t.* to water irrespective of the fact weather the water is stationary or flowing.

Motion of a Passenger Inside a Train

Suppose a train is moving with the velocity υ_t and a passenger moves with velocity υ_p inside the train. Two cases arise.

(1) If the passenger moves in the direction of motion of the train, then to the observer outside his (passenger's) velocity appears to be $\upsilon_t + \upsilon_p$.

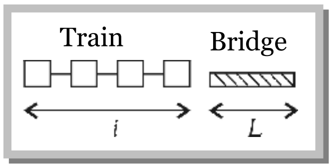
(2) If the passenger moves opposite to the direction of motion of the train, then to the observer outside the velocity of the passenger appears to be $\upsilon_t - \upsilon_p$.

Note : ❑ In the above case, we are not finding the relative velocity of the passenger *w.r.t* the train.

Train Crossing the Bridge

Suppose a train of length l is crossing a bridge of length L with constant speed υ. Then the time taken by the train to cross the bridge will be: $t = \dfrac{l + L}{v}$

Suppose the train accelerating at the rate a crosses a bridge. It enters the bridge with speed u. Then it will leave the bridge with speed υ, such that :



$$\upsilon^2 - u^2 = 2a(l + L) \quad \text{or} \quad \upsilon = \left[u^2 + 2a(l + L)\right]^{1/2}$$

To make relative velocity concept more clear consider the following examples:

Note : ❑ If speeds are comparable to the velocity of light c, according to theory of relativity, velocity of B relative to A (when both are moving along the same line in opposite directions) is given by : $\upsilon_{BA} = \dfrac{\upsilon_B + \upsilon_A}{[1 + (\upsilon_A \upsilon_B / c^2)]}$

From this, it is clear that if υ_A or υ_B is equal to c then $\upsilon_{BA} = \dfrac{\upsilon + c}{[1 + (\upsilon / c)]} = c$ *i.e.* speed of light is independent of relative motion between source and observer, the basic postulate of special theory of relativity.