LIMITS

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2. Limits

2.1 Limits

2.1.1 Concept of limits

Consider the function, $f(x)=\dfrac{x^2-4}{x-2}$ .We investigate the behaviour of f(x) at the point x=2 and near the point x=2.

Clearly, f (2) $=\dfrac{0}{0}$, which is meaningless. Thus f(x) when x is very near to 2 (and this will finally lead us to a value that would almost be the value of f(2)). Some value of f(x) for x less than 2 and then for x greater than 2 are given in the table below.

x<2		x>2	
x	f(x)	x	f(x)
1.9	3.9	2.1	4.1
1.95	3.95	2.05	4.05
1.99	3.99	2.01	4.01
1.995	3.995	2.005	4.005
1.999	3.999	2.001	4.001

It is clear from the table that as x gets nearer and nearer to 2 from either side; f(x) gets closer to 4 from either side. The number 4 is called the **limit** of f(x) as x approaches 2 and we write

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=4$$

(The symbol '$\rightarrow$' stands for approaches to')

Thus, we are led to the following definition of limit. If there exists a real number l such that if $\left|f(x)-l\right|$ can be made as small as possible by taking x sufficiently close to a, then l is called the limit of f (x) as x tends to 'a'. This is represented as $\lim\limits_{x\to a} f(x)=l$.

2.1.2 Definition of limit

We say that $\lim\limits_{x\to a} f(x)=l$, if the value of f(x) gets closer and closer to the number l as x gets nearer and nearer to a but not equal to a. For finding $\lim\limits_{x\to a}$f(x) we study the behavior of the function f in the neighbour of a and not at a. Thus, the function f may or may not be defined at x=a.

2.1.2.1 Left-hand Limit (LHL)

We say that left-hand limit of f(x) as x tends to 'a' exists and is equal to l_1 if as x approaches 'a' through values less than 'a', the values f(x) approach a definite unique real number l_1 and we write $\lim_{x\to a^-} f(x) = l_1$ or $f(a-0) = l_1$

Working Rule

To evaluate $\lim_{x\to a^-}$ f(x)

- Put x=a-h in f(x) to get $\lim_{h\to 0} f(a-h)$.
- Take the limit as h→0.

2.1.2.2 Right-hand Limit (RHL)

We say that right-hand limit of f(x) as x tends to 'a' exists and is equal to l_2 if as x approaches 'a' through values greater than 'a', the values of f(x) approach a definite unique real number l_2 and we write $\lim_{x\to a^+} f(x) = l_2 \;\; or \;\; f(a+0) = l_2$.

Working Rule

To evaluate $\lim_{x\to a^+}$ f(x)

- Put x=a+h in f(x) to get $\lim_{h\to 0}$ f(a+h)
- Take the limit as h→0

Note:

- A number is said to be a limiting value only if it is finite and real, otherwise we say that the limit does not exist.
- Individually the right hand limit and left hand limit are called **one-sided limits**.
- A limit is said to exist only if LHL and RHL are equal i.e., $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

Illustration: For the function $f(x) = x + (x-[x])^2$, find the RHL and LHL at x = 2.Check whether $\lim_{x\to 2} f(x)$ exists.

Solution: **Right hand limit =** $\lim_{x\to 2^+} f(x)$

Put x=2+h in f(x)

$$\Rightarrow \lim_{h\to 0} f\left(2+h\right) = \lim_{h\to 0}\left(2+h\right)+\left(2+h-\left[2+h\right]\right)^2$$

$$=\lim_{h\to 0}\left(2+h\right)+\left(2+h-2\right)^2$$

$=\lim_{h\to 0} 2+h+h^2$ ($\because$ [2+h] is the largest integer less than 2+h)

= 2

Left hand limit = $\lim_{x\to 2^-} f(x)$

Put x=2–h in f(x)

$$\Rightarrow \lim_{h\to 0} f(2-h) = \lim_{h\to 0}(2-h) + (2-h-[2-h])^2$$
$$= \lim_{h\to 0}(2-h) + (2-h-1)^2 = \lim_{h\to 0}(2-h) + (1-h)^2$$
$= 3$

Since LHL $\neq$ RHL $\Rightarrow \lim_{x\to 2} f(x)$ **does not exist.**

2.1.3 Algebra of limits

If $\lim_{x\to a}$ f(x) $= l$ and $\lim_{x\to a} g(x) = m$, then following results are true:

1. $\lim_{x\to a}[f(x) \pm g(x)] = \left(\lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)\right) = l \pm m$
2. $\lim_{x\to a} c.f(x) = c.\lim_{x\to a} f(x) = cl$, Where c is a constant.
3. $\lim_{x\to a} [f(x).g(x)] = \lim_{x\to a} f(x). \lim_{x\to a} g(x) = lm$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$ (provided $m \neq 0$)
5. $\lim_{x\to a}(fog)(x) = \lim_{x\to a} f[g(x)] = f\left(\lim_{x\to a} g(x)\right)$ =f (m)

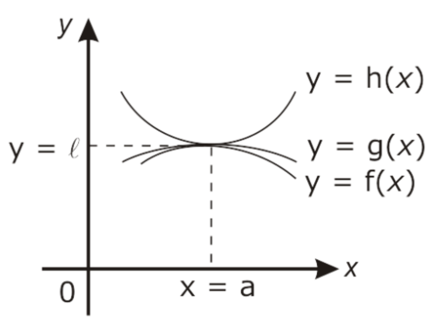
   In particular,

   - $\lim_{x\to a} \log g(x) = \log\left(\lim_{x\to a} g(x)\right) = \log m$
   - $\lim_{x\to a} e^{g(x)} = e^{\lim_{x\to a} g(x)} = e^m$

6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = l^n$, for all $n \in N$

7. **Sandwich Theorem** (or Squeeze Principle)

   If f, g and h are functions such that $f(x) \leq g(x) \leq h(x)$ for all x in some neighbourhood of the point a (except possibly at x=a) and if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.



SANDWICH THEOREM

On the graph there are 3 different functions f, g and h. The function whose limit we really want is g which is sandwiched between f and h. Since the limits of f and h exist at a and are equal then g is "forced" to have the same limit as it is sandwiched between f and h.

This theorem helps in calculating the limits, when limits cannot be calculated using the usual, method.

Illustration: If $1 \le f(x) \le x^2 + 2x + 2$ for all x, evaluate $\lim_{x \to -1} f(x)$

Solution: Again we have three functions f, g and h defined for all x. By applying Sandwich theorem we have

$$\lim_{x \to -1} 1 \le \lim_{x \to -1} f(x) \le \lim_{x \to -1} x^2 + 2x + 2$$

$$\Rightarrow 1 \le \lim_{x \to -1} f(x) \le \left((-1)^2 + 2(-1) + 2\right)$$

$$= 1 \le \lim_{x \to -1} f(x) \le 1$$

Observe that limits on either side of $\lim_{x \to -1} f(x)$ evaluated to 1.Since f is squeezed between the other two functions , which have the same limit at a=-1.Then by sandwich theorem we have $\lim_{x \to -1} f(x) = 1$.

8. If $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| = |l|$

The converse of the above result may not be true, i.e, $\lim_{x \to a} \left| f(x) \right| = |l| \ne \lim_{x \to a} f(x) = l$

2.1.4 Evaluation of limits

The problems on limits can be divided into the following categories:
(i) Algebraic limits
(ii) Trigonometric limits
(iii) Exponential and logarithmic limits

2.1.4.1 Algebraic limits

The following methods are useful for evaluating limits of algebraic functions:

a) Substitution

We can directly substitute the number at which limit is to be find.

Illustration: Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x} + \sqrt{1-x}}{1+x}$

Solution: We have,

$$\lim_{x \to 0} \frac{\sqrt{1+x} + \sqrt{1-x}}{1+x} = \frac{\sqrt{1+0} + \sqrt{1-0}}{1+0} = \frac{1+1}{1} = 2$$

b) Method Factorization

If f(x) and g(x) are polynomials and g(a) $\ne$ 0, then we have $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)}$

Now, if f(a)=0=g(a), then (x-a) is a factor of both f(x) and g(x). We cancel this common factor (x-a) from both the numerator and denominator and again, put x=a in the given expression. If we get a meaningful number then that number is limit of the given expression, otherwise we repeat this process till we get a meaningful number.

Illustration: Find $\lim_{x\to 1}\frac{x^3-x^2\ln x+\ln x-1}{x^2-1}$

Solution: The given limit is $\lim_{x\to 1}\frac{(x^3-1)-(x^2-1)\ln x}{x^2-1}$

When we substitute x=1 the denominator is zero.
Now we try for factorization method.

$$\Rightarrow \lim_{x\to 1}\frac{(x-1)(x^2+x+1)-(x-1)(x+1)\ln x}{(x-1)(x+1)}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x-1)[x^2+x+1-(x+1)\ln x]}{(x-1)(x+1)}$$

$$\Rightarrow \frac{1^2+1+1-(1+1)\ln 1}{(1+1)}$$

$$\Rightarrow \frac{3-0}{2}=\frac{3}{2}$$

c) Method of Rationalization

This method is useful where radical signs (i.e., expressions of the form $\sqrt{a}\pm\sqrt{b}$) are involved either in the numerator or in the denominator or both. The numerator or (and) the denominator (as required) is (are) rationalized and limit is taken after cancelling out the common factors.

Illustration: Find $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+x}}{x}$

Solution: The given limit is $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+x}}{x}$

$$=\lim_{x\to 0}\left(\frac{\sqrt{1+x^2}-\sqrt{1+x}}{x}\right)\left(\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^2}+\sqrt{1+x}}\right) \text{(By rationalization)}$$

$$=\lim_{x\to 0}\frac{\left(1+x^2\right)-\left(1+x\right)}{x\left(\sqrt{1+x^2}+\sqrt{1+x}\right)}=\lim_{x\to 0}\frac{x^2-x}{x\left(\sqrt{1+x^2}+\sqrt{1+x}\right)}$$

$$=\lim_{x\to 0}\frac{x\left(x-1\right)}{x\left(\sqrt{1+x^2}+\sqrt{1+x}\right)}=\lim_{x\to 0}\frac{\left(x-1\right)}{\left(\sqrt{1+x^2}+\sqrt{1+x}\right)}$$

$$=\frac{-1}{\sqrt{1}+\sqrt{1}}=-\frac{1}{2}$$

d) By using Standard Formula

$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, where $n \in Q$, the set of rational numbers.

Illustration: Evaluate $\lim_{x \to 1} \frac{x^{\frac{1}{3}} - 1}{x^{\frac{1}{2}} - 1}$

Solution: $\lim_{x \to 1} \frac{x^{\frac{1}{3}} - 1}{x^{\frac{1}{2}} - 1} = \lim_{x \to 1} \left(\frac{x^{\frac{1}{3}} - 1}{x - 1} \right) \left(\frac{x - 1}{x^{\frac{1}{2}} - 1} \right)$

$$= \left(\frac{\frac{1}{3} \times 1^{\frac{-2}{3}}}{\frac{1}{2} \times 1^{\frac{-1}{2}}} \right) = \frac{2}{3}$$

e) Evaluation of Algebraic Limit when x→∞

To find the limit of a function of the type $\frac{f(x)}{g(x)}$ as $x \to \infty$, where f(x) and g(x) are algebraic functions of x, it is convenient to divide all the terms of f(x) and g(x) by the highest power of x in numerator and denominator and use the following standard limits:

(**i**) $\lim_{x \to \infty} \frac{1}{x} = 0$ (**ii**) $\lim_{x \to \infty} \frac{1}{x^p} = 0$, if $p > 0$

- $$\lim_{n \to \infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } -1 < a < 1 \\ \text{does not exist}, & \text{if } a \le -1 \end{cases}$$

- $$\lim_{x \to \infty} \frac{a_0 x^p + a_1 x^{p-1} + \ldots \ldots + a_{p-1} x + a_p}{b_0 x^q + b_1 x^{q-1} + \ldots \ldots + b_{q-1} x + b_q} = \begin{cases} \frac{a_0}{b_0}, & \text{if } p = q \\ 0, & \text{if } p < q \\ \infty, & \text{if } p > q \end{cases}$$

Note: Write down the given expression in the form of a rational function if it is not so.

Illustration: Evaluate $\lim_{x \to \infty} \frac{(x+1)^{10} + (x+2)^{10} + \ldots. + (x+100)^{10}}{x^{10} + 10^{10}}$

Solution: We have, $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+....+(x+100)^{10}}{x^{10}+10^{10}}$

$$=\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+....+\left(1+\frac{100}{x}\right)^{10}}{1+\left(\frac{10}{x}\right)^{10}}$$

$$=\frac{1+1+....+1\ (100..times)}{1+0}=100$$

Illustration: Find $\lim_{x\to-\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)$

Solution: $\lim_{x\to-\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)$

$$=\lim_{x\to-\infty}\frac{\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}{\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}$$

$$=\lim_{x\to-\infty}\frac{\left(x^2+4x\right)-\left(x^2-4x\right)}{\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}=\lim_{x\to-\infty}\frac{8x}{\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}$$

Now divide the Nr and Dr with x, we have

$$=\lim_{x\to-\infty}\frac{8}{\left(\frac{\sqrt{x^2+4x}}{x}+\frac{\sqrt{x^2-4x}}{x}\right)}$$

Here x<0 $\therefore x=-\sqrt{x^2}$

$$\Rightarrow\lim_{x\to-\infty}\frac{8}{-\sqrt{\frac{x^2+4x}{x^2}}+\sqrt{\frac{x^2-4x}{x^2}}}=\lim_{x\to-\infty}\frac{8}{-\sqrt{1+\frac{4}{x}}-\sqrt{1-\frac{4}{x}}}$$

$$=\frac{8}{-1-1}=\frac{8}{-2}=-4$$

2.1.4.2 Trigonometric limits

For finding the limits of trigonometric functions, we use trigonometric transformations and simplify. The following results are quite useful.

1. (i) $\lim_{x\to0}\frac{\sin x}{x}=1$ (ii) $\lim_{x\to0}\cos x=1$

   (iii) $\lim_{x\to0}\frac{\tan x}{x}=1$ (iv) $\lim_{x\to0}\frac{\sin^{-1}x}{x}=1$

(v) $\lim_{x\to 0}\frac{\tan^{-1} x}{x}=1$ (vi) $\lim_{x\to 0}\frac{\sin x^0}{x}=\frac{\pi}{180}$

(vii) $\lim_{x\to 0}\frac{1\text{-}\cos\ x}{x}=0$

2. $\lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h),\ \text{where a}\neq 0\text{, on taking x}=\text{a+h}$

Some Useful Trigonometrical Expansions

1. $\sin\text{x}=\text{x}-\frac{\text{x}^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...\infty; x\in R$

2. $\cos\text{x}=1-\frac{\text{x}^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...\infty; x\in R$

3. $\tan\ \text{x}=\text{x}+\frac{\text{x}^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+...\infty;\quad |x|<\frac{\pi}{2}$

4. $\sec\ \text{x}=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+...\infty;\ \ 0<|x|<\frac{\pi}{2}$

5. $\text{cosec x}=\frac{1}{\text{x}}+\frac{x}{6}+\frac{7}{360}x^3+\frac{31}{15120}x^5+...\infty; 0<|x|<\pi$

6. $\cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2}{945}x^5-...\infty; 0\,|x|<\pi$

7. $\sin^{-1}x=x+\frac{1}{2}\ \frac{\text{x}^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5x^7}{2.4.6.7}+\infty;|x|<1$

8. $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$

$$=\frac{\pi}{2}-\left\{x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+...\infty\right\};|x|<1$$

9. $\tan^{-1}x=\begin{cases}\text{x}-\frac{\text{x}^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...\infty; & |x|<1\\ \pm\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\frac{1}{7x^7}-...\infty; & \begin{cases}+\ \text{if x}\geq 1\\ -\ \text{if x}\leq -1\end{cases}\end{cases}$

Illustration: Prove that $\lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}=-4$

Solution: We have, $\lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos(x+\pi/4)}$

$$= \lim_{x\to\pi/4} \frac{\tan x(\tan x-1)(\tan x+1)}{\cos(x+\pi/4)}$$

$$= \lim_{x\to\pi/4} \frac{\tan x(\sin x-\cos x)(\tan x+1)}{\cos x\ \cos(x+\pi/4)}$$

$$= -\lim_{x\to\pi/4} \frac{\tan x(\cos x-\sin x)(\tan x+1)}{\cos x\ \cos(x+\pi/4)}$$

$$= -\sqrt{2}\lim_{x\to\pi/4} \frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x.-\frac{1}{\sqrt{2}}\sin x\right)(\tan x+1)}{\cos x\ \cos(x+\pi/4)}$$

$$= -\sqrt{2}\lim_{x\to\pi/4} \frac{\tan x\cos(x+\pi/4).(\tan x+1)}{\cos x\ \cos(x+\pi/4)}$$

$$= -\sqrt{2}\lim_{x\to\pi/4} \frac{\tan x(\tan x+1)}{\cos x}$$

$$= -\sqrt{2}\times 2\sqrt{2} = -4.$$

2.1.4.3 Exponential and Logarithmic limits

For finding the limits of exponential and logarithmic functions, the following results are useful.

(i) $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

(ii) $\lim_{x\to 0} \frac{a^x-1}{x} = \log_e a\ ,\ a>0$

(iii) $\lim_{x\to 0} \frac{a^x-b^x}{x} = \log_e\left(\frac{a}{b}\right);\ a,\ b>0$

(iv) $\lim_{x\to 0} \frac{(1+x)^n-1}{x} = n$

(v) $\lim_{x\to 0}(1+x)^{1/x} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

(vi) $\lim_{h\to 0}(1+ah)^{1/h} = e^a$

(vii) $\lim_{x\to\infty}\frac{\log x}{x^m} = 0, (m>0)$

(viii) $\lim_{x\to\infty}\frac{\log_a(1+x)}{x} = \log_a e,\ (a>0,\ a\neq 1)$

(ix) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

(x) $\lim_{x\to a}(1+f(x))^{1/f(x)} = e$

(xi) $\lim_{x\to\infty}\left(1+\frac{1}{f(x)}\right)^{f(x)} = e$, where f(x) $\to\infty$ as x $\to\infty$

Some Useful Expansions

(i) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\text{to }\infty$

(ii) $e^{-x} = 1+\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\text{ to }\infty$

(iii) $\log\ (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...\ to\ \infty,\ -1\ < x \le 1$

(iv) $\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ...\ to\ \infty,\ -1 \le x < 1$

(v) $a^x = e^{x \log a} = 1 + x \log a + \frac{(x \log a)^2}{2!} + ...\ to\ \infty$

(vi) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + ...\ to\ \infty,\ -1 < x < 1,$ n being any negative integer or faction.

The expansion formulae mentioned above can be used with advantage in simplification and evaluation of limits.

Illustration: Show that $\lim\limits_{x \to 0} \frac{e - (1+x)^{1/x}}{x} = \frac{e}{2}$

Solution: $\lim\limits_{x \to 0} \frac{e - (1+x)^{1/x}}{x}$

$$= \lim_{x \to 0} \frac{e - e^{\frac{\ln(1+x)}{x}}}{x} = \lim_{x \to 0} -e \left(\frac{e^{\frac{\ln(1+x) - x}{x}} - 1}{x} \right)$$

Put $\frac{\ln(1+x) - x}{x} = y \quad \Rightarrow \lim\limits_{x \to 0} -e \left(\frac{e^y - 1}{y} \right) \left(\frac{\ln(1+x) - x}{x^2} \right)$

Now we have, $\lim\limits_{x \to 0} \frac{\ln(1+x) - x}{x} \quad \left(\frac{0}{0}\ form \right)$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x} - 1}{1} = 0$$

Also $\lim\limits_{x \to 0} \left(\frac{\ln(1+x) - x}{x^2} \right) \quad \left(\frac{0}{0}\ form \right)$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x} - 1}{2x} = \lim_{x \to 0} \frac{-1}{2(1+x)} = \frac{-1}{2}$$

Hence the required limit is $-e \lim\limits_{y \to 0} \left(\frac{e^y - 1}{y} \right) \lim\limits_{x \to 0} \left(\frac{\ln(1+x) - x}{x^2} \right)$

$$= -e.1.\frac{-1}{2} = \frac{e}{2}$$ (Hence proved)

Note:

- If $\lim_{x\to a} f(x) = A > 0$ and $\lim_{x\to a} g(x) = B$, then $\lim_{x\to a} [f(x)]^{g(x)} = A^B$
- If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$

2.1.5 Indeterminate forms

If a unique values cannot be assigned to f(a), then f(x) is said to be indeterminate at x=a.

Definition:

If $f(x)$ and $g(x)$ be any two functions of x, then we know that $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$. When $f(x)$ and $g(x)$ both tend to zero as $x \to a$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ when written in the form $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ reduces to the form $\frac{\mathbf{0}}{\mathbf{0}}$, which is meaningless.

A fraction whose numerator and denominator both tend to zero as $x \to a$ is called the **indeterminate form** $\frac{0}{0}$.It has no definite values. The other indeterminate forms are $\frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^{\infty}, 0^0, \infty^0$.

Note:

If a fraction $\frac{f(x)}{g(x)}$ takes the indeterminate form $\frac{0}{0}$. When $x \to a$ it does not mean that $\lim_{x\to a}\frac{f(x)}{g(x)}$ will not exist.

For example the fraction $\frac{x^2 - a^2}{x - a}$ takes the indeterminate form $\frac{0}{0}$ when $x \to a$ .But $\lim_{x\to a}\frac{x^2 - a^2}{x - a} = \lim_{x\to a}\frac{(x-a)\ (x+a)}{x-a} = \lim_{x\to a}\ (x+a) = 2a$, showing that the limit exists in this case. Thus $\lim_{x\to a}\frac{f(x)}{g(x)}$ may exist even if the fraction $\frac{f(x)}{g(x)}$ takes the indeterminate form $\frac{0}{0}$.

2.1.5.1 L' Hospital's rule

Besides the method discussed above to evaluate limits, there is yet another method for finding limits, usually known as L' Hospital's rule as given below for indeterminate forms:

$\left(\frac{\mathbf{0}}{\mathbf{0}}\right)$ **form**: If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S exists.

$\left(\frac{\infty}{\infty}\right)$ **form**: If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S exists.
Here, f' is derivative of f.

Note: Sometimes, we have to repeat the process if the form is $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ again.

Note:

- L' Hospital's Rule is applicable only when $\frac{f(x)}{g(x)}$ becomes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- If the form is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, simplify the given expression till it reduces the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L' Hospital's rule.
- For applying L' Hospital's rule differentiate the numerator and denominator separately.

$\left(\frac{0}{0}\right)$ Form:

Illustration: Evaluate $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

Solution: Here $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ [form 0/0 so we shall apply Hospital's rule]

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2} \quad \text{[form again 0/0]}$$

$$=\lim_{x\to 0}\frac{\sin x}{6x} \quad \text{[form again 0/0]}$$

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$

$\left(\frac{\infty}{\infty}\right)$ Form:

Illustration: Evaluate $\lim_{x\to\infty}\frac{x^2+2x}{5-3x^2}$

Solution: We have, $\lim_{x\to\infty}\frac{x^2+2x}{5-3x^2}$ [form ∞/∞]

$$=\lim_{x\to\infty}\frac{2x+2}{-6x}, \quad \text{[form } \infty/\infty\text{]}$$

$$=\lim_{x\to\infty}\frac{2}{-6}=-\frac{1}{3}.$$

$0\times\infty$ Form:

This form can be easily reduced to the form $\frac{0}{0}$ or to the form $\frac{\infty}{\infty}$. Let $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$ then $\lim_{x\to a} f(x)\, g(x)=\lim_{x\to a}\frac{f(x)}{1/g(x)}$ (form $\frac{0}{0}$)

$$\text{or } =\lim_{x\to a}\frac{g(x)}{1/f(x)} \text{ (form } \frac{\infty}{\infty})$$

Thus $\lim_{x\to a} f(x)\, g(x)$ is reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which can now be evaluated by L' Hospital's rule.

Illustration: Evaluate $\lim_{x\to 1}(1-x).\tan\frac{\pi x}{2}$

Solution: We have $\lim_{x\to 1}(1-x).\tan\frac{\pi x}{2}$, [form $0\times\infty$]

$=\lim_{x\to 1}\frac{1-x}{\cot(\pi x/2)}$, [form $0/0$]

$$=\lim_{x\to 1}\frac{-1}{-\frac{\pi}{2}.\operatorname{cosec}^2\frac{\pi x}{2}}=\lim_{x\to 1}\frac{2}{\pi}\sin^2\frac{\pi x}{2}=\frac{2}{\pi}.$$

$\infty-\infty$ Form:

When $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$ then

$$\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a}\left[\frac{1}{1/f(x)}-\frac{1}{1/g(x)}\right] \qquad or =\lim_{x\to a}\left[\frac{1/g(x)-1/f(x)}{1/f(x)\,1/g(x)}\right] \text{(form } \frac{0}{0})$$

which can now be evaluated by using L' Hospital's rule.

Illustration: Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

Solution: We have $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ [form $\infty-\infty$]

$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}$, [form $0/0$]

$$=\left(\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4}\right).\left(\lim_{x\to 0}\frac{x^2}{\sin^2 x}\right),$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4}, \qquad \left[\because \lim_{x\to 0} \frac{x^2}{\sin^2 x} = 1\right]$$

$$= \lim_{x\to 0} \frac{1}{x^4}\left[\left(x - \frac{x^3}{3!} + \ldots\right)^2 - x^2\right]$$

$$= \lim_{x\to 0} \frac{1}{x^4}\left[\left\{x^2 - \frac{1}{3}x^4 + \ldots\right\} - x^2\right]$$

$$= \lim_{x\to 0}\left[-\frac{1}{3} + \text{terms containing } x \text{ and its higher powers}\right]$$

$$= -\frac{1}{3}.$$

$1^{\infty}, 0^{0}, \infty^{0}$ Forms:

Suppose $\lim_{x\to a}\left[f(x)\right]^{g(x)}$ takes any one of these forms. Then let $y = \lim_{x\to a}\left[f(x)\right]^{g(x)}$

Taking logarithm of both sides, we get $\log y = \lim_{x\to a} g(x)\cdot \log f(x)$

Now in any of the above three cases log y takes the form $0\times\infty$ which is changed to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ whichever is convenient and then its limit is evaluated by L' Hospital's rule or by using standard expressions.

Illustration: Evaluate $\lim_{x\to \pi/2} (\sin x)^{\tan x}$

Solution: Let $y = \lim_{x\to \pi/2} (\sin x)^{\tan x}$ [form 1^{∞}]

$\therefore\ \log y = \lim_{x\to \pi/2} \tan x.\ \log \sin x,$ [form $\infty\times 0$]

$= \lim_{x\to \pi/2} \frac{\log\ \sin x}{\cot\ x},$ [form $0/0$]

$= \lim_{x\to \pi/2} \frac{(1/\sin\ x)\cos\ x}{-\text{cosec}^2 x},$ [by L' Hospital's rule]

$= \lim_{x\to \pi/2} (-\sin x\ \cos x) = 0.$

$\therefore\ y = e^0 = 1$

Illustration: Evaluate $\lim_{x\to 0}\ (\text{cosec}\ x)^{1/\log x}$

Solution: Let $y = \lim_{x\to 0}\ (\text{cosec}\ x)^{1/\log x}$ [form ∞^0]

$$\therefore \ \log y = \lim_{x\to 0} \frac{1}{\log x}(\log \ \operatorname{cosec} x) \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x\to 0} \frac{(1/\operatorname{cosec} x)\ (-\operatorname{cosec} x \ \cot x)}{1/x}$$

$$= \lim_{x\to 0} \left(\frac{-x}{\tan x}\right), \qquad \text{[form 0/0]}$$

$$= \lim_{x\to 0} \left(\frac{-1}{\sec^2 x}\right) = -1$$

$$\therefore \ y = e^{-1} = 1/e.$$

2.1.6 Solved Examples

Example: Evaluate $\lim_{x\to\infty} \frac{3x+\sin 2x}{3x-\sin 2x}$

Solution: $\lim_{x\to\infty} \frac{3x+\sin 2x}{3x-\sin 2x}$

$$= \lim_{x\to\infty} \frac{3+2\left(\frac{\sin 2x}{2x}\right)}{3-2\left(\frac{\sin 2x}{2x}\right)}$$

$$= \frac{3+2(0)}{3-2(0)} = 1, \text{ since } \lim_{x\to\infty} \frac{\sin 2x}{2x} = 0.$$

Example: Evaluate $\lim_{x\to 1} \frac{x^{p+1}-(p+1)x+p}{(x-1)^2}$

Solution: $\Rightarrow \lim_{x\to 1} \frac{x^{p+1}-(p+1)x+p}{(x-1)^2}$

$$\Rightarrow \lim_{x\to 1} \frac{x^{p+1}-px-x+p}{(x-1)^2}$$

$$\Rightarrow \lim_{x\to 1} \frac{x(x^p-1)-p(x-1)}{(x-1)^2}$$

Dividing N^r and D^r by (x-1) we get the limit

$$\Rightarrow \lim_{x\to 1} \frac{x\frac{(x^p-1)}{(x-1)}-p}{(x-1)}$$

$$\Rightarrow \lim_{x\to 1}\frac{x+x^2+x^3+\dots\ \dots+x^p-p}{(x-1)}$$

(Using summation of x terms of G.P.)

$$\Rightarrow \lim_{x\to 1}\frac{(x-1)+(x^2-1)+(x^3-1)+\dots..\ +(x^p-1)}{(x-1)}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x-1)}{(x-1)}+\frac{(x^2-1)}{(x-1)}+\frac{(x^3-1)}{(x-1)}+\ \dots\ \dots\ \dots+\frac{(x^p-1)}{(x-1)}$$

$$\Rightarrow\ 1+2+3+\ \dots\ \dots\ \dots\ \dots\ \dots+p$$

$$\Rightarrow \frac{p(p+1)}{2}$$

Example: *Evaluate* $\lim_{x\to\pi/4}\frac{1-\tan x}{1-\sqrt{2}\sin x}$

Solution: $\lim_{x\to\pi/4}\frac{1-\tan x}{1-\sqrt{2}\sin x}$

$$\Rightarrow \lim_{x\to\pi/4}\left(1-\frac{\sin x}{\cos x}\right)\frac{1}{\sqrt{2}(1/\sqrt{2}-\sin x)}$$

$$\Rightarrow \lim_{x\to\pi/4}\frac{1}{\cos x}\left(\frac{\cos x-\sin x}{\sqrt{2}}\right)\frac{1}{(\sin\pi/4-\sin x)}$$

$$\Rightarrow \lim_{x\to\pi/4}\frac{1}{\cos x}(\sin\pi/4\ \text{cosx-cos}\pi/4\ \text{sinx})\frac{1}{(\sin\pi/4-\sin x)}$$

$$\Rightarrow \lim_{x\to\pi/4}\frac{1}{\cos x}\times\frac{\sin(\pi/4-x)}{(\sin\pi/4-\sin x)}$$

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$$= \lim_{x\to\pi/4}\frac{1}{\cos x}\times\frac{2\sin\left(\frac{\pi/4-x}{2}\right)\cos\left(\frac{\pi/4-x}{2}\right)}{2\cos\left(\frac{\pi/4+x}{2}\right)\sin\left(\frac{\pi/4-x}{2}\right)}\Rightarrow\ 2$$

Example: *Evaluate* $\lim_{x\to 0}\frac{a^x-1}{x}$

Solution: $\lim_{x\to 0}\frac{a^x-1}{x}$

$\because\ a^x=e^{x\ln a}$

The given limit $\lim_{x\to 0}\frac{e^{x\ln a}-1}{x}$

$$\Rightarrow \lim_{x\to 0}\frac{\left(1+x\ln a+\frac{(x\ln a)^2}{2!}+\dots\ \dots to\ \infty\right)-1}{x}$$

$$\Rightarrow \lim_{x}\frac{x\ln a}{x}\left(1+\frac{x\ln a}{2!}+terms\text{ of higher power of x}\right)$$

$$\Rightarrow\ \ln a$$

or, $\lim_{x\to 0}\frac{e^{x\ln a}-1}{x}$

$$\Rightarrow \lim_{x\to 0}\left(\frac{e^{x\ln a}}{x\ln a}\right)\times\ \ln a \Rightarrow 1\times\ln a \Rightarrow\ \ln a$$

Example: $\lim_{n\to\infty}\frac{(n+2)!\ +\ (n+1)!}{(n+2)!\ -\ (n+1)!}$

Solution: Since (n+2)! = (n+2)(n+1)!

Hence the given limit $\Rightarrow \lim_{n\to\infty}\frac{(n+2)(n+1)!+(n+1)!}{(n+2)(n+1)!-(n+1)!}$

$$\Rightarrow \lim_{n\to\infty}\frac{(n+2+1)(n+1)!}{(n+2-1)(n+1)!}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n(1+3/n)}{n(1+1/n)} \Rightarrow\ 1$$

Example: $\lim_{x\to\pm\infty}\frac{\sqrt{x^2+x}}{x+6}$

Solution: $\frac{\sqrt{x^2+x}}{x+6}$

$$\Rightarrow \frac{\sqrt{x^2+(1+1/x)}}{x(1+6/x)}$$

$$\Rightarrow \frac{\sqrt{x^2}\sqrt{1+1/x}}{x(1+6/x)}$$

$$\Rightarrow \frac{|x|\sqrt{1+1/x}}{x(1+6/x)}$$

When $\quad x \to +\infty;\ |x| = x$

Hence the given limit $\Rightarrow \lim\limits_{x\to+\infty} \dfrac{x\sqrt{1+1/x}}{x(1+6/x)} \Rightarrow 1$

When $x \to -\infty; |x| = -x$

Hence the given limit $\Rightarrow \lim\limits_{x\to-\infty} \dfrac{-x\sqrt{1+1/x}}{x(1+6/x)} \Rightarrow -1$

Example: Evaluate $\lim\limits_{x\to -\infty} \dfrac{x^4 \sin\frac{1}{x} + x^2}{1+|x|^3}$

Solution: $\lim\limits_{x\to -\infty} \dfrac{x^4 \sin\frac{1}{x} + x^2}{1+|x|^3}$

$= \lim\limits_{x\to -\infty} \dfrac{x^4 \sin\frac{1}{x} + x^2}{1-x^3}$ $\left[\because x \to -\infty \Rightarrow x < 0 \quad \therefore |x| = -x\right]$

$= \lim\limits_{x\to -\infty} \dfrac{\dfrac{x^4 \sin\frac{1}{x}}{\frac{1}{x}}\dfrac{1}{x} + x^2}{1-x^3} = \lim\limits_{x\to -\infty} \dfrac{\dfrac{\sin\frac{1}{x}}{\frac{1}{x}} + \dfrac{1}{x}}{\dfrac{1}{x^3} - 1}$

$= \dfrac{1+0}{0-1} = -1$

Example: Let f (x) be a function defined by f (x) = $\begin{cases} 4x-5, \ if \ x \le 2 \\ x-\lambda \ , if \ x > 2 \end{cases}$

Find λ, *if* $\lim\limits_{x\to 2} f(x)$ exists.

Solution: We have,

f (x) = $\begin{cases} 4x-5, \ if \ x \le 2 \\ x-\lambda \ , if \ x > 2 \end{cases}$

$\therefore \lim\limits_{x\to 2} f(x) = \lim\limits_{h\to 0} f(2-h)$

$= \lim\limits_{h\to 0} 4(2-h) - 5$

$= \lim\limits_{h\to 0} 3 - 4h = 3$ and,

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2-h)$$

$$= \lim_{h \to 0} 2 + h - \lambda = (2-\lambda)$$

If $\lim_{x \to 2} f(x)$ exists,then $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$\Rightarrow$ 3 = 2–λ

$\Rightarrow$ λ = –1.

Example: If f (x) = $\begin{cases} |x|+1, & x<0 \\ 0 & , x=0. \\ |x|-1, & x>0 \end{cases}$ For what value(s) of 'a' does $\lim_{x \to a} f(x)$ exist?

Solution: We have,

f(x) = $\begin{cases} |x|+1, & x<0 \\ 0 & , x=0. \\ |x|-1, & x>0 \end{cases}$

$\Rightarrow$ f(x) = $\begin{cases} -x+1, & x<0 \\ 0 & , x=0. \\ x-1, & x>0 \end{cases}$ $\left[\because |x| = \begin{cases} x & x \geq 0 \\ -x & x<0 \end{cases} \right]$

Clearly, $\lim_{x \to a} f(x)$ exists for all $a \neq 0.$

Now, check whether $\lim_{x \to 0} f(x)$ exist or not.

We have,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(0-h) = \lim_{h \to 0} -(-h)+1 = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h-1 = -1$$

$$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

So, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $\lim_{x \to a} f(x)$ exists for all $a \neq 0.$

Example: Evaluate $\lim_{x \to 1} \frac{(1-x+\ln x)}{(1+\cos \pi x)}$

Solution: $\lim_{x \to 1} \frac{(1-x+\ln x)}{(1+\cos \pi x)}$ is of the form $0/0$.

$$= \lim_{x \to 1} \frac{\left(-1+\frac{1}{x}\right)}{(-\pi \sin \pi x)} = \lim_{x \to 1} \frac{(-x+1)}{(x(-\pi \sin \pi x))}$$

$$= \lim_{x \to 1} \frac{(x-1)}{(\pi x \sin \pi x)}$$ (Algebraic simplification)

Still it is of the form 0/0. Again by applying L' Hospital's rule we have

$$= \lim_{x \to 1} \frac{1}{\left(\pi \sin \pi x + \left(\pi^2 x \cos \pi x\right)\right)}$$

$$= \frac{-1}{\pi^2} \qquad (\because \ \sin \pi = 0 \ and \ \cos \pi = -1)$$

Example: Evaluate $\lim_{x \to 0} \frac{[e^x + \log\{(1-x)/e\}]}{\tan x - x}$.

Solution:

$$\lim_{x \to 0} \frac{e^x + \log\left(\frac{1-x}{e}\right)}{\tan\ x - x} = \lim_{x \to 0} \frac{e^x + \log\ (1-\text{x}) - \log \text{e}}{\tan x - x}$$

$$= \lim_{x \to 0} \frac{e^x + \log\left(\frac{1-x}{e}\right) - 1}{\tan\ x - x},$$ [form 0/0]

$$= \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right) - 1}{\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\right) - x}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{6} x^3 (1 + \text{terms containing } x \text{ and its higher powers})}{\frac{1}{3} x^3 (1 + \text{terms containing } x \text{ and its higher powers})}$$

$$= \frac{-1}{2}$$