COMPLEX NUMBERS

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- An ordered pair (a,b) of real numbers a, b is called a complex number and is written as $z = a + ib$
- **Iota:** The symbol, I, iota, is defined as the solution of the equation $x^2 + 1 = 0$
  The symbol I is used to denote the number $\sqrt{-1}$.
- **Equality of Complex Numbers:**
  Two complex numbers (a, b) and (c, d) are said to be equal if and only if $a = c$ and $b = d$.

- **Addition of Complex Numbers:** If $z_1 = (a, b)$ and $z_2 = (c, d)$ be two complex numbers, then their sum $z_1 + z_2$ is defined as the complex number
  $(a + c, b + d)$. Thus $z_1 + z_2 = (a, b) + (c, d) = (a + c, b + d)$.

- **Properties of Addition:**

(i) Addition is commutative, i.e.,
$(a, b) + (c, d) = (c, d) + (a, b)$

(ii) Addition is associative, i.e.,
$[(a, b) + (c, d)]+(e, f) = (a, b)+[(c, d)+(e, f)]$

(iii) Existence of zero, i.e., there exists the complex number (0, 0) such that
$(a, b) + (0, 0) = (a, b) \ \forall \ (a, b) \in Z.$

(iv) Existence of negative, i.e., for each complex number (a, b) there exists its negative (-a, -b).

- **Difference of Complex Numbers:** If z_1 = (a, b) and z_2 = (c, d) be two complex numbers, then their difference z_1-z_2 is defined as the complex number
(a-c, b-d)

- **Product of Complex Numbers:** If z_1 = (a, b) and z_2 = (c, d) be two complex numbers, then their product $z_1.z_2$ is defined as the complex number (ac - bd, ad + bd). The operation of multiplication satisfies the following properties:

(i) Multiplication is commutative, i.e.,
(a, b) . (c, d) = (c, d). (a, b)

(ii) Multiplication is associative, i.e.,
[(a, b).(c, d)].(e, f)=(a, b).[(c, d). (e, f)]

(iii) Existence of unity, i.e., there exists the complex number (1, 0) called the identity such that

(a, b).(1, 0) = [(a, 1 - b.0)(a.0 + b.1)]

= (a, b) $\forall$ (a, b) $\in$ Z

(iv) Existence of inverse, i.e., to every complex number z = (a, b), other than (0, 0), there exists the complex number z_1 = (c, d) such that

(a, b).(c, d)= (1, 0)

$$\text{where } c = \frac{a}{a^2+b^2}, d = \frac{-b}{a^2+b^2}$$

This inverse z_1 of z is denoted by z^{-1}.

- **Division of Two complex Numbers:** If z_1 and z_2 be any two complex numbers, $z_2 \neq 0$, then their division is defined by $z_1 \div z_2 = z_1.z_2^{-1}$
  The division $z_1 \div z_2$ is denoted by $\frac{z_1}{z_2}$
- **Real and Imaginary Parts of a Complex Number:** If z = a + ib be any complex number, then a and b are called the real and the imaginary parts of z respectively.
- **Complex Conjugate:** The complex conjugate of a complex number z = a +ib is defined as a – ib, denoted by $\overline{z}$, and vice-versa.
  Some of the properties of conjugation are:

  $(i) \quad \overline{(\overline{z})} = z$ $\qquad$ $(ii) \quad z + \overline{z} = 2\,\mathrm{Re}(z)$

(iii) $z - \bar{z} = 2i \operatorname{Im}(z)$

(iv) $z.\bar{z} = a^2 + b^2$, Where z = a + ib

(v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

vi) $\overline{Z_1.Z_2} = \overline{Z_1}.\overline{Z_2}$

(vii) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$, provided $Z_2 \neq 0$

- **Some Properties of Complex Numbers:**

Now, let us consider some interesting properties of conjugates.

✓ Conjugate of the conjugate of a complex number z is the complex number itself, i.e., $\overline{(\bar{z})} = z$

✓ The conjugate of the sum of the two complex numbers z_1, z_2 is the sum of their conjugates, i.e., $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

✓ The conjugate of the product of two complex numbers z_1, z_2 is the product of their conjugates, i.e., $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

- ✓ The conjugate of the quotient of two complex numbers z_1, z_2 ($z_2 \neq 0$) is the quotient of their conjugates, i.e., $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

  Let us now, consider some properties of absolute values of complex numbers.

- ✓ The absolute value of a product of two complex numbers z_1 and z_2 is equal to the product of the absolute values of the numbers, i.e., $|z_1 z_2| = |z_1||z_2|$.

- ✓ The absolute value of a quotient of two complex numbers z_1 and z_2 ($\neq 0$), is equal to the quotient of the absolute values of the dividend and the divisor, i.e.,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0.$$

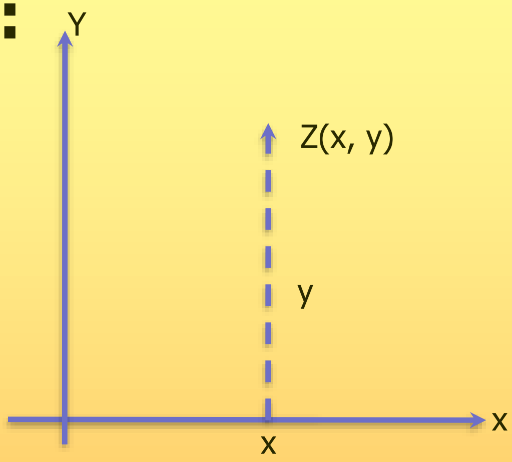
- ✓ The absolute value of the sum of two complex numbers z_1 , z_2 can never exceed the sum of their absolute values, i.e.,
  $| z_1 + z_2| \leq | z_1|+ |z_2|$.
  This inequality is called *triangle inequality*.
  The triangle inequality can be extended to *n* complex numbers by finite induction, *i.e.*, for any *n* complex numbers z_1 , z_2 ……., z_n, we obtain
  $| z_1 + z_2 +\ldots\ldots.+ z_n| \leq | z_1|+ |z_2|+ \ldots\ldots.+|z_n|$.

- ✓ The absolute value of the difference of two complex numbers z_1 and z_2 can never be less than the difference of their absolute values,
  i.e., $| z_1 - z_2| \geq | z_1|- |z_2|$.

- **Representation of Complex Numbers:**

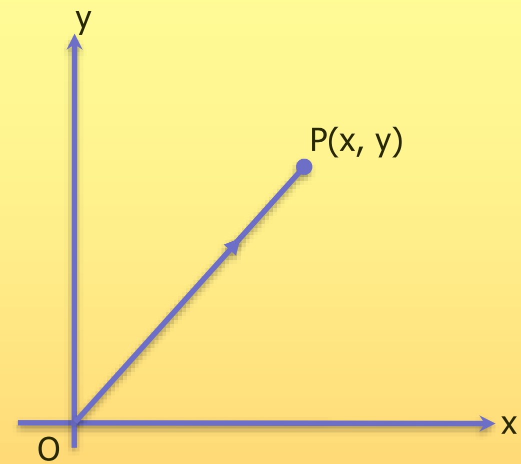
✓ *Geometrical Representation* : Let OX, OY be the set of axes. Let $z = (x + iy)$ be any given complex number.

Now associate a point Z to be complex number z in the Cartesian plane, whose coordinates are (x, y). The point Z is then called the image of the complex number z.



Conversely, also to each point Z′ (x′, y′) in the plane is the image a complex number $z' = x' + iy'$. Thus, there is one to one correspondence between the set of all complex numbers and the set of points on an plane. This representative plane is called **Argand plane**.

✓ *Vector Representation* : Let P (x, y) represent the complex number z = x + iy, then the vector $\overrightarrow{OP}$ denotes the complex number z. Also z is called the position vector of P, i.e., $\overrightarrow{OP}$.



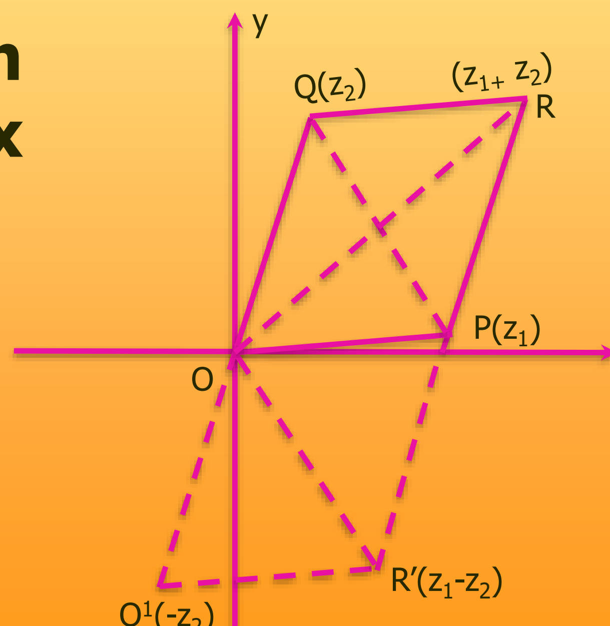
Vector Representation of Addition and Subtraction of Complex Number

Let P be $z_1 = x_1 + i\,y_1$ and

Q be $z_2 = x_2 + i\,y_2$, then

$\overrightarrow{OP} = x_1 + iy_1 = z_1,$

$\overrightarrow{OQ} = x_2 + iy_2 = z_2.$



$\therefore\ z_1 - z_2 = \overrightarrow{QP}$

Let Q′ be – z_2, i.e., $(-x_2, -y_2)$

then completing the parallegram OQ′R′P, we have

$$\overrightarrow{OR'} = \overrightarrow{OP} + \overrightarrow{OQ'} = z_1 + (-z_2) = z_1 - z_2,$$

then R′ = $z_1 - z_2$

- **Trigonometric Representation**

  Let z = a + ib be a complex number.

  Write a = r cos θ, b = r sin θ, where r, θ are real numbers and r is non-negative.

  Then

$$z = a + ib = r(\cos\theta + i\sin\theta)$$

$$\text{Also } r = +\sqrt{a^2 + b^2}$$

Also cos θ + i sin θ is $e^{i\theta}$ (*Euler's Formula*)

- **Modulus:** r is called the *modulus* or *absolute value* of the complex number *z*, and is denoted by $|z|$. So

$$|z| = \sqrt{a^2 + b^2}$$

Argument: θ is called the *argument* or *amplitude* of the complex number *z* and is denoted by $\theta = \text{Arg } z$.

The value of θ, where $-\pi < \theta \leq \pi$ is called the principal value of the argument.

Properties of Modulus Value

- ✓ $|z| = 0 \ \forall \ z = 0$
- ✓ $z.\bar{z} = |z|^2$
- ✓ $|z_1 z_2| = |z_1| \ |z_2|$
- ✓ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$

- ✓ $|z_1 + z_2| \leq |z_1| + |z_2|$
- ✓ $||z_1| - |z_2|| \leq |z_1 - z_2|$
- ✓ $-|z| \leq \text{Re}(z) \leq |z|$
- ✓ $-|z| \leq \text{Im}(z) \leq |z|$
- ✓ $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Properties of Argument

- ✓ $\text{Arg.}(z_1 z_2) = \text{Arg.}\,Z_1 + \text{Arg.}\,Z_2$
- ✓ $\text{Arg.}\left(\frac{z_1}{z_2}\right) = \text{Arg.}\,z_1 - \text{Arg.}\,z_2$

Observations

- ➢ The amplitude of a positive real number is zero.
- ➢ The amplitude of a negative real number is π
- ➢ The amplitude of a positive imaginary number is $\frac{\pi}{2}$.
- ➢ The amplitude of a negative imaginary number is $\frac{-\pi}{2}$ or $\frac{3\pi}{2}$.

Cube roots of unity: Let z denote a cube root of unity. Then

$z = (1)^{1/3}$ or $z^3 = 1$

or $(z-1)(z^2 + z + 1) = 0$

$\therefore$ $z-1 = 0$ or $z^2 + z + 1 = 0 \Rightarrow z = 1$

$$\text{or } z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Hence, the cube roots of unity are

$$1, \omega = \frac{-1 + i\sqrt{3}}{2}, \omega^2 = \frac{-1 - i\sqrt{3}}{2}.$$

so that $1 + \omega + \omega^2 = 0$.

- **De Moivre's Theorem**

- If *n* is an integer, positive or negative, the *n* according to De Moivre's Theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

- If n is not an integer, then one of the values of $(\cos\theta + \sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Determination of roots of a complex number: If n is a positive integer, then $(\cos\theta + i\sin\theta)^{1/n}$ has exactly n distinct roots. Given by

$$\cos\frac{2k\pi+\theta}{n}, i\sin\frac{2k\pi+\theta}{n}$$

for k = 0, 1, 2, ……., n -1

Expansion of cos nθ and sin nθ in terms of power of cos θ and sin θ:

$\cos n\theta = \cos^n \theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \ldots$;

and

$\sin n\theta = {}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5 \theta + \ldots$

Thank You…