Limit

Limit of a Function

Let $y = f(x)$ be a function of x. If at $x = a, f(x)$ takes indeterminate form, then we consider the values of the function which are very near to 'a'. If these values tend to a definite unique number as x tends to 'a', then the unique number so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \to a} f(x)$.

(1) **Meaning of '$x \to a$' :** Let x be a variable and a be the constant. If x assumes values nearer and nearer to 'a' then we say 'x tends to a' and we write '$x \to a$'. It should be noted that as $x \to a$ we have $x \neq a$. By 'x tends to a' we mean that

(i) $x \neq a$ (ii) x assumes values nearer and nearer to 'a' and

(iii)We are not specifying any manner in which x should approach to 'a'. x may approach to a from left or right as shown in figure.

(2) **Left hand and right hand limit :** Consider the values of the functions at the points which are very near to a on the left of a. If these values tend to a definite unique number as x tends to a, then the unique number so obtained is called left-hand limit of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$.

Similarly we can define right-hand limit of $f(x)$ at $x = a$ which is expressed as $f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$.

(3) **Method for finding L.H.L. and R.H.L. :**

(i) For finding right hand limit (R.H.L.) of the function, we write $x + h$ in place of x, while for left hand limit (L.H.L.) we write $x - h$ in place of x.

(ii) Then we replace x by 'a' in the function so obtained.

(iii) Lastly we find limit $h \to 0$.

(4) **Existence of limit :** $\lim_{x \to a} f(x)$ exists when

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.* L.H.L. and R.H.L. both exists. (ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.* L.H.L. = R.H.L.

Note : ❑ If a function $f(x)$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then we say that $f(x)$ is indeterminate or meaningless at $x = a$. Other indeterminate forms are $\infty - \infty, \infty \times \infty, 0 \times \infty, 1^{\infty}, 0^0, \infty^0$

❑ In short, we write L.H.L. for left hand limit and R.H.L. for right hand limit.

❑ It is not necessary that if the value of a function at some point exists then its limit at that point must exist.

Fundamental theorems on limits

The following theorems are very useful for evaluation of limits if $\lim_{x\to 0} f(x) = l$ and $\lim_{x\to 0} g(x) = m$ (l and m are real numbers) then

(1) $\lim_{x\to a}(f(x)+g(x)) = l+m$ (Sum rule)

(2) $\lim_{x\to a}(f(x)-g(x)) = l-m$ (Difference rule)

(3) $\lim_{x\to a}(f(x).g(x)) = l.m$ (Product rule)

(4) $\lim_{x\to a} k\ f(x) = k.l$ (Constant multiple rule)

(5) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m}, m \neq 0$ (Quotient rule)

(6) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a}\frac{1}{f(x)} = 0$

(7) $\lim_{x\to a}\log\{f(x)\} = \log\left\{\lim_{x\to a} f(x)\right\}$

(8) If $f(x) \leq g(x)$ for all x, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$

(9) $\lim_{x\to a}[f(x)]^{g(x)} = \{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(10) If p and q are integers, then $\lim_{x\to a}(f(x))^{p/q} = l^{p/q}$, provided $(l)^{p/q}$ is a real number.

(11) If $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(m)$ provided 'f' is continuous at $g(x) = m$. *e.g.* $\lim_{x\to a}\ln[f(x)] = \ln(l)$, only if $l > 0$.

Some important expansions

In finding limits, use of expansions of following functions are useful –

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

(2) $a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + ...$

(3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...,|\ x| < 1$

(5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4},......,|\ x| < 1$

(6) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

(7) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....$

(8) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(9) $\sin h\,x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

(10) $\cos h\,x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$

(11) $\tan h\,x = x - \frac{x^3}{3} + 2x^5 - .....$

(12) $\sin^{-1} x = x + 1^2.\frac{x^3}{3!} + 3^2.1^2.\frac{x^5}{5!} + ...$

(13) $\cos^{-1} x = (\pi/2) - \sin^{-1} x$

(14) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

Methods of evaluation of limits

We shall divide the problems of evaluation of limits in five categories.

(1) **Algebraic limits :** Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim_{x\to a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method :** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) **Factorisation method :** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalisation method :** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

(iv) **Based on the form when $x \to \infty$:** In this case expression should be expressed as a function $1/x$ and then after removing indeterminant form, (if it is there) replace $1/x$ by 0.

Step I : Write down the expression in the form of rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.

Step II : If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k.

Step III : Use the result $\lim_{x \to \infty} \frac{1}{x^n} = 0$, where $n > 0$.

Note : ❑ **An important result :** If m, n are positive integers and $a_0, b_0 \neq 0$ are non-zero real numbers, then $\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + ... + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + ... + b_{n-1} x + b_n} = \begin{cases} \frac{a_0}{b_0}, \text{ if } m = n \\ 0, \text{ if } m < n \\ \infty, \text{if } m > n \end{cases}$.

(2) **Trigonometric limits :** To evaluate trigonometric limits the following results are very important.

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x}$ (ii) $\lim_{x \to 0} \frac{\tan x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan x}$ (iii) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin^{-1} x}$

(iv) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan^{-1} x}$ (v) $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$ (vi) $\lim_{x \to 0} \cos x = 1$

(vii) $\lim_{x \to a} \frac{\sin(x-a)}{x-a} = 1$ (viii) $\lim_{x \to a} \frac{\tan(x-a)}{x-a} = 1$ (ix) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a, |a| \leq 1$

(x) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a; |a| \leq 1$ (xi) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a; -\infty < a < \infty$

(xii) $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} = 0$ (xiii) $\lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$

(3) **Logarithmic limits :** To evaluate the logarithmic limits we use following formulae

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....\text{to } \infty$ where $-1 < x \leq 1$ and expansion is true only if base is e.

(ii) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$ (iii) $\lim_{x \to e} \log_e x = 1$ (iv) $\lim_{x \to 0} \frac{\log(1-x)}{x} = -1$

(4) **Exponential limits :** (i) Based on series expansion – We use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$

To evaluate the exponential limits we use the following results –

(a) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ (b) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$ (c) $\lim_{x \to 0} \frac{e^{\lambda x} - 1}{x} = \lambda \ (\lambda \neq 0)$

(ii) **Based on the form 1^∞ –** To evaluate the exponential form 1^∞ we use the following results :–

(a) If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$, or when $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$.

Then $\lim_{x \to a} \{f(x)\}^{g(x)} = \lim_{x \to a} [1 + f(x) - 1]^{g(x)} = e^{\lim_{x \to a} (f(x)-1)g(x)}$.

(b) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (c) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(d) $\lim_{x \to 0} (1 + \lambda x)^{1/x} = e^\lambda$ (e) $\lim_{x \to \infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda$

Note : ❑ $\lim_{x \to \infty} a^x = \begin{cases} \infty & \text{if } a > 1 \\ 0 & \text{if } a < 1 \end{cases}$ *i.e.*, $a^\infty = \infty$ if $a > 1$ and $a^\infty = 0$ if $a < 1$.

(5) **L – Hospital's rule :** If $f(x)$ and $g(x)$ be two functions of *x* such that

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ (ii) Both are continuous at $x = a$

(iii) Both are differentiable at $x = a$

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$

Note : ❑ The above rule is also applicable if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$

❑ If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L – Hospital rule, we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$. Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.