THREE DIMENSIONAL GEOMETRY

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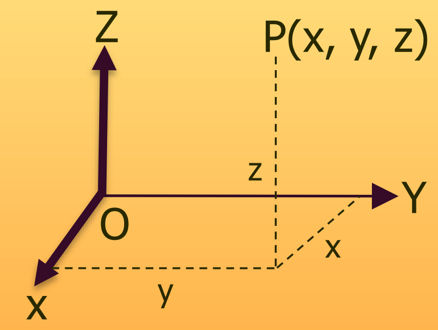
- **Orthogonal Cartesian co-ordinates:** The position of a point P in a space is characterised by three Cartesian co-ordinates, x, y, z i.e., P(x, y, z).



- **Distance between two points:** The distance between the point P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) is

$$\overline{P_1 P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

PLANES

A geometrical locus is a plane if it is such that if P and Q are any two points on the locus, then every point of the line PQ is also a point on the locus.

Result : Every equation of the first degree in x, y, z represents a plane.

- **Equation of plane:** The equation of every plane is of the first degree, i.e., it is of the form $ax + by + cz + d = 0$.
  in which a, b, c are constants where $a^2 + b^2 + c^2 \neq 0$
- **One-point form:** Equation of any plane passing through (x_1, y_1, z_1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0.$$

where A, b, C are constants.

- **Intercept form of a plane:** The equation of a plane in terms of the intercepts a, b, c which it make with the axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

- **Normal form of a plane:** Let *p* be the length of the perpendicular form the origin to the plane and l, m, n be the direction cosines of this perpendicular.
Then the equation of the plane is
lx + my + nz = p.

Note: If the equation of the plane is given in general form, i.e., ax + by + cz + d = 0, then a, b, c represent the direction ratios of the normal to the plane, provided d is negative.

- **Angle between the two planes:** Angle between the two planes is the angle between the normals to them.
  The angle formed by the two planes
  $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$
  is given by

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\sum a_1^{2}} \times \sqrt{\sum a_2^{2}}}\right|, \theta \leq 90^\circ$$

- **Condition of parallelism between planes:** The two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel of $\theta = 0$, i.e., $\cos\theta = 1$

$$\text{or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- **Condition of orthogonality between planes:** The two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are orthogonal iff $\theta = 90^\circ$, i.e., $\cos \theta = 0$.
  Therefore, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

- **Distance of a plane from the origin:** Given the plane $ax + by + cz + d = 0$, its distance from the origin of the axes is given by

$$\left| \frac{d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

- **Length of the perpendicular from a point to a plane:** The perpendicular distance of the point P (x_1, y_1, z_1) from the plane $lx + my + nz = p$ is
  $lx_1 + my_1 + nz_1 - p$

The length of the perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$\pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$

- **Equation of any plane parallel to the given plane:** Equation of any plane parallel to the plane $ax + by + cz + d = 0$ is
  $ax + by + cz + k = 0$
  Rule: Replace the constant term by k and keep the other terms as they are in the given equation.

- **Equation of the plane through the intersection of two planes (System of planes):** Equation of any plane through the intersection of two planes
  $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is
  $a_1x + b_1y + c_1z + d_1 + k\,[a_2x + b_2y + c_2z + d_2] = 0$, k being a parameter

- **Bisectors of angles between two planes:** The equations of the bisectors of angles between the planes $ax + by + cz + d = 0$, $a_1x + b_1y + c_1z + d_1 = 0$ are

$$\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

STRAIGHT LINES (3D)

A straight line in space is characterised by the intersection of two planes which are not parallel and, therefore, the equation of a straight line is present as a solution of the system constituted by the equation of the two planes:

$$a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0$$

This form is also known as *unsymmetrical form.*

- **Symmetrical Forms**

✓ *One-point form*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

are the equation of the line which passes through the point A (x_1, y_1, z_1) and whose direction cosines are l, m, n. Also

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

are the equation of the line which passes through (x_1, y_1, z_1) and whose direction cosines are proportional to a, b, c.

✓ *Two-point form*

Equations of the line which passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

- **Angle between a line and a plane:** The angle between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$ is $\theta=\sin^{-1}\left[\frac{al+bm+cn}{\sqrt{l^2+m^2+n^2}\sqrt{a^2+b^2+c^2}}\right]$

  where l, m, n are direction ratios of the line.

- **Condition of parallelism:** The line is parallel if $\theta = 0$, i.e., $al + bm + cn = 0$.

- **Condition in order that the line may lie on the given plane:** The line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

  will lie on the plane $Ax + By + Cz + D = 0$ if

(i) $Al + Bm + Cn = 0$ (ii) $Ax_1 + By_1 + Cz_1 + D = 0$

- **Condition of perpendicularity:** The line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ will be perpendicular to the plane Ax + By + Cz + D = 0 if $\frac{l}{A}=\frac{m}{B}=\frac{n}{C}$

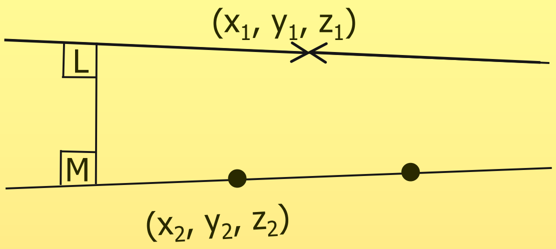
- **Equation of any plane through a given line:** Equation of any plane through the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ provided $Al+Bm+Cn=0.$

- **Equation of plane through the intersecting lines:** Equation of any plane through the intersecting lines

$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=o$

- **Skew lines:** Two straight lines are said to be the skew if they do not intersect and are not parallel as well, i.e., they do not lie in the same plane.



- **Line of the shortest distance:** The straight line which is prependicular to each of the two skew lines is called the *shortest distance* and the length of this line intercepted between the given lines (i.e., LM in the figure) is called the length of the shortest distance.

SPHERES

A sphere is the locus of a point which remains at a constant distance from a fixed point.
The constant distance is called the **radius** and the fixed point on the **centre** of the sphere.

Equation of a sphere:

Let (a, b, c) be the centre and r the radius of a given sphere. Then
the equation of the sphere is
$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

- **General equation of a sphere:**
  The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre at (-u, -v, -w) and

  $$\text{Radius} = \sqrt{u^2 + v^2 + w^2 - d}$$

  where u is half the coefficient of x, v is half the coefficient of y, w is half the coefficient of z, d is the constant term, provided coefficients of x^2, y^2, z^2 are unity.

- **Plane section of a sphere:** *Plane section of a sphere is a circle. The centre of the circle is the foot N of the perpendicular from the centre O of the sphere to the plane and its radius is* $\sqrt{r^2 - ON^2}$, *r being the radius of the sphere.*

- **Great circle:** The section of a sphere by a plane through its centre is a known as a *great circle.*
- **Intersection of two spheres:** The locus of the points of intersection of two spheres is a circle.
- **Sphere with a given diameter:** The equation of the sphere described on the segment joining points A (x_1, y_1, z_1) B (x_2, y_2, z_2) as a diameter is
  $(x-x_1)(x-x_2)+(y-y_2)+(z-z_1)(z-z_2) = 0$
- **Equation of a circle:** A circle can be represented by two equations, one representing a sphere and the other a plane.
  Hence, equations of circle are
  $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, $P \equiv Ax + By + Cz + D = 0$
  A circle can also be represented by the equations of any two spheres through it. Thus, a circle may also be given by the equation $S_1 = 0$, $S_2 = 0$.

- **Spheres through a given circle:**
  The equations $S + \lambda P = 0$ represents spheres through the circle with equations $S = 0$, $P = 0$,
  where $S = x^2 + y^2 + z^2 + 2uz + 2vy + 2wz + d = 0$,
  $P \equiv lx + my + nz - p = 0$
  Also the equation $S + \lambda S' = 0$ represents a sphere through the circle with equation $S = 0$, $S' = 0$, where
  $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$
  $S' = x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d'$
  for all values of λ.
- **Equation of a tangent:** The equation of the tangent plane at any point (α, β, γ) on the sphere
  $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is
  $(\alpha + u)\, x + (\beta + v)\, y + (\gamma + w)\, z + (u\alpha + v\beta + w\gamma + d) = 0.$

- **Plane of contact:** The locus of points of contact of the tangent planes which pass through a given point (α, β, γ) and touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $x(\alpha + u) + y(\beta + v) + z(\gamma+w) + (u\alpha + v\beta + w\gamma + d)= 0.$

- **Angle of intersection of two spheres:** The angle of intersection of the sphere at a common point is the angle between the tangent planes to them at that point and is, therefore, also equal to the angle between the radii of the spheres to the common point .

- **Orthogonal sphere:** The spheres are said to be *orthogonal* if angle of intersection of spheres is a right angle.

- **Condition for the orthogonality of two spheres:** The condition for two spheres
$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$
$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$
to be orthogonal is $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

- **Radical plane:** The radical plane of two spheres $S_1 = 0, S_2 = 0.$ in both of which the coefficients of the second degree terms are equal to unity, is $S_1 – S_2 = 0$

- **Radical centre:** The radical centre of four spheres
$S_1 = 0, S_2 = 0, S_3 = 0, s_4 = 0$ is the intersection of the two lines
$S_1 - S_2 = 0, S_2 – S_3 = 0, S_1 – S_3 = 0, S_2 – S_4 = 0$

- **Co- axial systems:** A system of spheres, any two members of which have the same radical plane, is called a co-axial system of spheres. The system of spheres $S_1 + \lambda S_2 = 0$ is co-axial.

- **Limiting points:** The limiting points of co-axial system are the points spheres of the system, i.e., spheres of radii zero.

Limiting points of the sphere, $x^2 + y^2 + z^2 + 2kx + d = 0$

$are(-\sqrt{d},0,0),(\sqrt{d},0,0).$

Thank You…