Permutation and Combination

Permutation

Definition

A permutation of a set of different things is an arrangement of these things taken some or all at a time

For *example :* Three different things a, b and c are given, then different arrangements which can be made by taking two things from three given things are ab, ac, bc, ba, ca, cb .

There fore the number of permutations will be 6.

Notation : If *n* and *r* are positive integer such that $1 \le r \le n$. Then the number of all permutation of *n* distinct things, taken *r* at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$

$$i.e. \quad {}^nP_r = n(n-1)(n-2)\text{.........} \text{..........} \text{..........} \text{....}(n-(r-1))$$

$${}^nP_r = \frac{n!}{(n-r)!}$$

Permutation under certain conditions

(1) The number of all permutations of *n* different objects taken *r* at a time, when a particular object is to be always included in each arrangement is $r\,{}^{n-1}P_{r-1}$.

(2) The number of permutations of *n* distinct objects taken *r* at a time, when a particular object is never taken in each arrangement is ${}^{n-1}P_r$.

(3) The number of permutations of *n* different objects taken *r* at a time in which two specified object always occur together is $2!(r-1)\,{}^{n-2}P_{r-2}$.

Permutation of objects not all distinct

The number of mutually distinguishable permutation of *n* things, taken all at a time, of which *p* are alike of one kind, *q* alike of second such that $p + q = n$ is $\frac{n!}{p!q!}$.

Note : ❑ The number of permutations of *n* things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind;............ p_r are alike of *r*th kind such that $p_1 + p_2 + ... + p_r = n,$ is $\frac{n!}{p_1!p_2!p_3!....p_r!}$.

❑ The number of permutations of *n* things, of which *p*. are alike of one kind, *q* are alike of second kind and remaining all are distinct is $\frac{n!}{p!q!}$.

❑ Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots\ldots\ldots .p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times etc. Then the total number of permutation of these r objects to the above condition is $\frac{(p_1 + p_2 + \ldots\ldots\ldots + p_r)!}{p_1! p_2! p_3! \ldots\ldots\ldots p_r!}$.

Permutation when objects can repeat : The number of permutation of n different things, taken r at a time when each may be repeated any number of times in each arrangement, is n^r.

Circular permutation

The number of circular permutation of n distinct objects is $(n-1)!$

Note : ❑ In the above statement anti- clockwise and clock wise order of arrangement are considered as distinct permutations.

If anti-clockwise and clockwise order of arrangement are not distinct e.g. arrangement of beads in a necklace, arrangement of flowers in a garland etc. Then the number of circular permutation of n distinct items is $\frac{1}{2}\{(n-1)!\}$.

Combination

Definition

A group or selection which can be made by taking some or all of a given number of things, without any regard to the order in which they are selected, is called combination of these things.

Example **:** All combinations of four letter A, B, C, D taken two at a time are AB, AC, AD, BC, BD, CD

Note : ❑ Generally we use the word 'arrangements' for permutation and the word 'selections' for combination.

Notation : The number of all combinations of n objects, taken r at a time is generally denoted by $C(n, r)$ or $\binom{n}{r}$ or ${}^nC_r$,

${}^nC_r$ or $C(n, r)$ = Number of ways of selecting r objects from n objects.

${}^nC_r$ is defined only when n and r are non-negative integers such that $0 \le r \le n$.

Combination of *n* different things taken *r* at a time : The number of all combinations of n distinct objects, taken r at a time is given by ${}^nC_r = \frac{n!}{(n-r)!r!}$.

Selection of one or more items

(1) **Selection from different items :** The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

(2) **Selection from identical items :** (i) The number of ways of selecting r items out of n identical items is 1.

(ii) The total number of ways of selecting zero or more i.e. at least one item from a group of n identical items is $(n+1)$.

(iii) The total number of selections of some or all out of $p+q+r$ items where p are alike of one kind, q are a like of second kind and rest r are alike of third kind is $[(p+1)\,(q+1)\,(r+1)-1]$.

(3) **Selection of items from a group containing both identical and different items :**

The total number of ways of selecting one or more items from p identical items of one kind; q identical items of second kind; r identical items of third kind and n different items is $(p+1)\,(q+1)\,(r+1)2^n - 1$.

Division of items into groups

(1) **Division of items into groups of unequal size :** Number of ways in which $(m+n)$ items can be divided into two unequal groups containing m and n items is $\frac{(m+n)!}{m!n!}$.

Note : ❑ The number of ways in which $(m+n+p)$ items can be divided into unequal group containing m,n,p items is ${}^{m+n+p}C_m \cdot {}^{n+p}C_n = \frac{(m+n+p)!}{m!n!p!}$.

❑ The number of ways to distribute $(m+n+p)$ items among 3 persons in the groups containing m,n and p items is = (Number of ways to divide) × (Number of groups) ! = $\frac{(m+n+p)!}{m!n!p!} \times 3!$

(2) **Division of items into groups of equal size :** The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the group is not important is $\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}$.

The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of group is important is $\left(\frac{(mn)!}{(n!)^m} \times \frac{1}{m!}\right)m! = \frac{(mn)!}{(n!)^m}$.

Division of identical objects into groups

(1) The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0,1,2 or more items $(\le n)$ is ${}^{n+r-1}C_{r-1}$. or

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

(2) The total number of ways of dividing n identical items among r persons, each one of whom, receives at least one item is ${}^{n-1}C_{r-1}$. or

The number of ways in which n identical items can be divided into r groups such that blank group are not allowed is ${}^{n-1}C_{r-1}$.

Some important results of permutation and combination

(1) ${}^nP_r = n\,{}^{n-1}P_{r-1}$

(2) ${}^nP_r = (n-r+1)\,{}^nP_{r-1}$

(3) ${}^nP_n = {}^nP_{n-1}$

(4) ${}^nP_r = {}^nC_r \times r!$

(5) ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

(6) ${}^nC_r = {}^nC_{n-r}$

(7) ${}^nC_p = {}^nC_q \Rightarrow p+q=n$ or $p=q$

(8) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(9) $^nP_r + r.\,^nP_{r-1} = \,^{n+1}P_r$

(10) $^nC_r = \frac{n-r+1}{r}\,^nC_{r-1}$

(11) $^nC_0 + \,^nC_1 + \,^nC_2 + ... + \,^nC_n = 2^n$

(12) $^nC_0 + \,^nC_2 + \,^nC_4 + ... = \,^nC_1 + \,^nC_3 + \,^nC_5 + ... = 2^{n-1}$

(13) Greatest value of $^nC_r = \,^nC_{n/2}$ when n is even.

$= \,^nC_{(n-1)/2}$ or $^nC_{(n+1)/2}$ when n is odd.

Some important results

(1) Given n distinct points in the plane, no three of which are collinear then the number of line segments formed $= \,^nC_2$, but if m of these points are collinear $(m \geq 3)$ then the number of line segment is $(\,^nC_2 - \,^mC_2) + 1$.

In particular the number of diagonals in an n sided closed polygon $= \,^nC_2 - n$

(2) Given n distinct points in the plane, no three of which are collinear. Then the number of triangles formed is $^nC_3$. If m of these points are collinear $(m \geq 3)$ the number of triangles formed $= \,^nC_3 - \,^mC_3$

(3) Given n points on the circumference of a circle, then

(i) Number of straight lines = $^nC_2$

(ii) Number of triangles = $^nC_3$

(iii) Number of quadrilaterals $= \,^nC_4$.

Hints and tricks

(1) The sum of n digit numbers formed by using n non zero digits without repetition = $(n-1)!$ (sum of digits) (111 .... n times) and sum of the digits in the unit place of such numbers $= (n-1)!$ (sum of digits).

(2) If given n digit include 0, then total number of n digit numbers formed with them = $^nP_m - \,^{n-1}P_{m-1}$. here $^{n-1}P_{m-1}$ is the number of such numbers which contains 0 at their extreme left.

(3) The number of ways in which m (one type of different) things and n (another type of different) things can be arranged in a row so that all the second type of things come together $= n!(m+1)\,!$.

(4) The number of ways in which n (one type of different) things and $n-1$ (another type of different) things can be arranged in a row so that no two thing of same type come together $= n!(n-1)\,!$

(5) The number of ways in which m (one type of different) things and n (another type of different) things $(m \geq n)$ can be arranged in a circle so that no two things of second kind come together $= (m-1)!\,^mP_n$ and when things of second type come together = $m\,!n\,!$.