DEFINITE INTEGRAL

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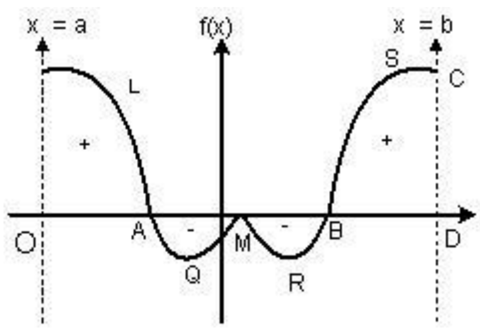
2.1 Geometrical Interpretation of Definite Integral

If f(x) > 0 for all x ∈ [a, b]; then $\int_a^b f(x)$ is numerically equal to the area bounded by the curve y = f(x), then x-axis and the straight lines x = a and x = b i.e. $\int_a^b f(x)$



In general $\int_a^b f(x)\,dx$ represents to algebraic sum of the figures bounded by the curve

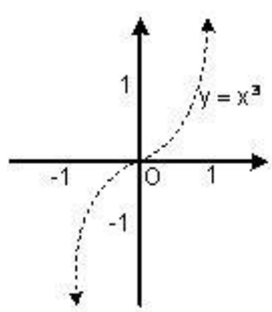
y = f(x), the x-axis and the straight line x = a and x = b. The areas above x-axis are taken place plus sign and the areas below x-axis are taken with minus sign i.e,

i.e. $\int_a^b f(x)\,dx$ area OLA – area AQM – area MRB + area BSCD

Note: $\int_a^b f(x)\,dx$, represents algebraic sum of areas means, that if area of function y = f(x) is asked between a to b.

=> Area bounded = $\int_a^b |f(x)|dx$ and not been represented by $\int_a^b f(x)\,dx$

e.g., If some one asks the area of $y = x^3$ between -1 to 1.



Then $y = x^3$ could be plotted as;

∴ Area = $\int_{-1}^{0} -x^3\,dx + \int_0^1 x^3\,dx = 1/2$

or, using above definition Area = $\int_{-1}^{1} |x^3|\,dx = 2\int_0^1 x^3\,dx$

$= 2\ [x^4 / 4]^1_0 = 1/2$

But if, we integrate x^3 between -1 to 1.

=> $\int^1_0 x^3\ dx = 0$ which does not represent area.

Thus, students are adviced to make difference between area and definite Integral.

2.2 Fundamental Theorem Of Calculus(Newton-Leibnitz Formula)

This theorem state that If f(x) is a continuous function on [a, b] and F(x) is any anti derivative of f(x) on [a, b] i.e. $F'(x) = f(x)\ \forall\ x \in (a, b)$, then $O'^b_a f(x)dx = F(b) - F(a)$

The function F(x) is the integral of f(x) and a and b are the lower and the upper limits of integration.

Illustration : *Evaluate $\int^2_{-2} dx / 4 + x^2$ directly as well as by the substitution x = 1 /t. Examine as to why the answer do not tally?*

Solution: $I = \int^2_{-2} dx / 4 + x^2$

$= [1/2 \tan^{-1} (x/2)]^2_{-2} = 1/2\ [\tan^{-1}(1) - \tan^{-1} (-1)]$

$= 1/2\ [\pi/4 - (-\pi/4)] = \pi/4 \Rightarrow I = \pi/4$

On the other hand; if x = 1/t then,

$I = \int^2_{-2} dx / 4 + x^2 = \int^{1/2}_{-1/2} dt / t^2 (4 + 1 / t^2) = \int^{1/2}_{-1/2} dt/ 4t^2 + 1$

$= -[1/2 \tan^{-1} (2t)]^{1/2}_{-1/2}$

$= -1/2 \tan^{-1} - (-1/2 \tan^{-1} (-1) = -\pi/8 - \pi/8 = -\pi/4$

$\therefore I = \pi/4$ when x = 1/t

In above two results $I = -\pi/4$ is wrong. Since the integrand $1/4+x^2 > 0$ and therefore the definite integral of this function cannot be negative.

Since x = 1/t is discontinuous at t = 0, the substitution is not valid ($\therefore I = \pi/4$).

Note: It is important the substitution must be continuous in the interval of integration.

Illustration : *Let* $\alpha = \int^\infty_0 dx / x^4 + 7x^2 + 1$ and $\beta = \int^\infty_0 x^2\ dx / x^4 + 7x^2 + 1$ *then show that* $\alpha = \beta$

Solution: $\alpha = \int^\infty_0 dx / x^4 + 7x^2 + 1$

put $x = 1/t \Rightarrow dx = -1/t^2$ then

$\alpha = \int^{\infty}_0 -/t^2\, dt / 1/t^4 + 7/t^2 + 1 = \int^{\infty}_0 dtt^2 / t^4 + 7t^2 + 1 = \int^{\infty}_0 t^2 dt / t^4 + 7t^2 + 1 = \beta$

2.3 Properties Of Definite Integration

1. Change of variable of integration is immaterial so long as limits of integration remain the same i.e. $\int^b_a f(x)dx = \int^b_a f(t)\, dt$

2. $\int^b_a f(x)dx = - \int^b_a f(x)dx$

3. $\int^b_a f(x)dx = \int^c_a f(x)dx + \int^b_a f(x)dx$

Generally we break the limit first at the points where f(x) is discontinuous and second at the points where definition of f(x) changes.

Illustration : *Evaluate* $\int^{5\pi/12}_0 [\tan x]dx$*, where [.] is the greatest integer function.*

Solution: Let $I = \int^{5\pi/12}_0 [\tan x]\, dx$

Value of tan x at x = 5π/12

Value of tan x at x = 0 is 0

Integers between 0 and $2 + \sqrt{3}$ are 1, 2, 3

$\therefore \tan x = 1, \tan x = 2, \tan x = 3$

$\Rightarrow x = \tan^{-1} 1, x = \tan^{-1} 2, x = \tan^{-1} 3$

$\therefore I = + \int_{\tan-10} [\tan x]\, dx + \int^{\tan-1\,2}_{\tan^{-1}\,1} [\tan x]dx + \int^{\tan-13}_{\tan^{-1}\,2} [\tan x]\, dx + \int^{15\pi/12}_{\tan^{-1}\,3} [\tan x]dx$

$= \int_{\tan-10} 0\, dx + \int_{\tan-1\,2\tan^{-1}\,1} 1\, dx + \int_{\tan-1\,3\tan^{-1}\,2} 2\, dx + \int^{15\pi/12}_{\tan^{-1}\,3} 3dx$

$= 0 + (\tan^{-1} 2 - \tan^{-1} 1) + 2 (\tan^{-1} 3 - \tan^{-1} 2) + 3 (5\pi /12 - \tan^{-1} 3)$

$= 5\pi / 4 - \pi/4 - \tan^{-1} 3 - \tan^{-1} 2$

$= \pi = [\tan^{-1} (3 + 2 / 1 - 6) + \pi] = -\tan^{-1} (-1)$

$= \pi/4$

4. $\int^a_0 f(x)dx = \int^a_0 f(a-x)\, dx.$

Illustration : *If f, g, h be continuous function on [0, a] such that*

f(a - x) = f(x), g(a - x) = - g(x) and 3h(x) - 4h(a - x) = 5,

then prove that $\int^a_0 f(x)\, g(x)\, h(x)dx = 0.$

Solution: I =

= $-\int^a_0 f(x)\, g(x)\, h(a-x)\, dx$

$7I = 3I + 4I$

= $\int^a_0 f(x)\, g(x)\, \{3h(x)-4h(a-x)\}\, dx$

= $5\int^a_0 f(x)\, g(x)\, dx = 0$, since $f(a-x)\, g(a-x) = -f(x)\, g(x)$

=> $I = 0$

Illustration : $\int^\pi_0 x \sin 2x \sin(\pi / 2 \cos x)\, dx$

Solution: Let $I = \int^\pi_0 x \sin 2x \sin(\pi/2 \cos x)\, dx$ ....(1)

= $\int^\pi_0 (\pi - x) \sin 2(\pi - x) \sin(\pi/2 \cos(\pi - x))\, dx$

= $\int^\pi_0 (\pi - x)(-\sin 2x)(-\sin 2x) \sin(-\pi/2 \cos x)\, dx$

= $\int^\pi_0 (\pi - x) \sin 2x \sin(\pi/2 \cos x)\, dx$ ....(2)

Adding (1) & (2), we get

$2I = \pi \int^\pi_0 \sin 2x \sin(\pi/2 \cos x)\, dx$

=> $I = \pi \int^{\pi/2}_0 2 \sin x \cos x \sin(\pi/2 \cos x)\, dx$

Put $\pi/2 \cos x = z$ => $-\pi/2 \sin x\, dx = dz$

= $\int^0 \pi/2\, z \sin z\, sz = 8/\pi$.

5. $\int^a 0\, f(x)\, dx = \int^{a/2} 0\, [f(x) + f(a-x)]\, dx$

Special cases: If $f(x) = f(a-x)$, then $\int^a 0\, f(x)dx = 2 \int^{1/2} 0\, f(x)\, dx$.

If $f(x) = -f(a-x)$, then $\int^a 0\, f(x)\, dx = 0$.

6. $\int^a -a\, f(x)\, dx = \int^a 0\, [f(x0 + f(-x)]dx$

Special case: $\int^a -a\, f(x)\, dx = 2\int^a 0\, f(x)dx$, if f(x) is even, $\int^a -a\, f(x)\, dx = 2$, if f(x) is odd

Illustration : *Evaluate* $\int^4_{-4} x^2 / (x^2 + 16)(1+e^{x5})\, dx$

Solution: Let $I = \int^4_{-4} f(x) / (1 + e^{x5})\, dx$ $(f(x) = x^2 / x^2 + 16)$

$2I = \int^4_{-4} f(x) / (1 + e^{x5})\, dx + \int^4_{-4} f(x) / (1 + e^{x5})\, dx$

$= dx$

$= \int_{-4}^{4} f(x)\, dx$

$I = \int_{0}^{4} x^2 / x^2 + 16\, dx$

$= 4 - \tan^{-1}1$

Illustration : *Find the value of is* $\int_{0}^{\pi} dx / 1 + 5^{\cos x} + \int_{-2}^{2} \log (5 - x / 5 + x)\, dx$ is

Solution: Let $I = I_1 + I_2$

Consider $I_1 = \int_{0}^{\pi} dx / 1 + 5^{\cos x}$ ...(1)

Now $I_1 = \int_{0}^{\pi} dx / 1+ 5^{\cos(\pi-x)} = \int_{0}^{\pi} dx / 1+ 5^{-\cos x}$

$= \int_{0}^{\pi} 5^{\cos x}\, dx / 5^{\cos x} + 1$ ...(2)

Adding (1) and (2) , we get

$2I_1 = \int_{0}^{\pi} dx / 1+ 5^{\cos x} + \int_{0}^{\pi} 5^{\cos x}\, dx / 5^{\cos x} + 1 = \int_{0}^{\pi} 1.dx = \pi$

$I_1 = \pi/2$

Consider $I_2 = \int_{-2}^{2} \log (5-x / 5+x)\, dx$

Let $g(x) = \log (5-x / 5+x)$

Now $g(-x) = \log(5 - (x) / 5+x) = -\log 5-x/5+x = -g(x)$

$\therefore$ g(x) is an odd function

$\therefore \int_{-2}^{2} g(x)dx = 0 \Rightarrow I_2 = 0$

$I = I_1 + I_2 = \pi/2 + 0 = \pi/2$

7. $\int_{0}^{1} f((b-a)\, x + a)\, dx$

Illustration : ***Evaluate*** $\int_{1/3}^{2/3} e^{9(x-2/3)2}\, dx$

Solution: $I_1 = \int_{-4}^{-5} e^{(x+5)2}\, dx$

$= (5-4) \int_{0}^{1} e^{((-5+4)\, x-4+5)2}\, dx$

$\int_{0}^{1} e^{(x-1)2}\, dx$ ...(i)

Again let $I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)2}\,dx$

$I_2 = (2/3 - 1/3) \int_0^1 e^{9[(2/3 - 1/2)\,x + 1/3 - 2/3]2}\,dx$

$= 1/3 \int_0^1 e^{(x-1)2}\,dx$

where $I = I_1 + 3I_2$

$I_1 + 3(-I_1/3) = I_1 - I_1$

$I = 0$

$\int_{1/3}^{2/3} e^{9(x-2/3)2}\,dx = 0$

8. If f (x) is a periodic function with period T, then

$\int_a^{a+nT} f(x)dx = n \int_0^T f(x)\,dx$ where $n \in 1$,

In particular,

(i) if a = 0, $\int_0^{nT} f(x)\,dx = n \int_0^T f(x)\,dx$ where $n \in I$

(ii) If n = 1, $\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$

Illustration : *Evaluate* $\int_0^{10\pi} |\sin x|dx$.

Solution: Let $I = \int_0^{10\pi} |\sin x|dx$

We know that |sinx| is a periodic function with period p

Hence $I = 10 \int_0^{\pi} |\sin x|\,dx$ [applying prop. 8]

Illustration : *If f(x) is a function satisfying f(x + a) + f(x) = 0 for all x ∈ R and constant a such that* $\int_b^{c+b} f(x)$ *dx is independent of b, then find the least positive value of c.*

Solution: We have f(x + a) + f(x) for all $x \in R$ ……(i)

=> f(x + a + a) + f(x + a) = 0 [Replacing x by x + a] ….(ii)

=> f(x + 2a) + f(x+ a) = 0 ….(iii)

Subtracting (i) from (ii), we get f(x + 2a) - f(x) = 0 for all $x \in R$.

=> f(x + 2a) = f(x) for all $x \in R$

So, f(x) is periodic with period 2a

It is given that $\int^{c+b}_{b} f(x)dx$ is independent of b.

∴ The minimum value of 'c' is equal to the period of f(x) i.e., 2a.

2.4 Differentiation Under The Integral Sign

Leibnitz's Rule

If g is continuous on [a, b] and f_1 (x) and f_2 (x) are differentiable functions whose values lie in [a, b], then d/dx $\int^{f_2(x)}_{f1(x)} g(t)dt = g(f_2(x))f_2'(x) – g(f_1(x))f_1'(x)$

Illustration : *If f(x) = cos x - $\int^{x}_{0}$ (x–t) then show that f'' (x) + f (x) = – cosx*

Solution: $f'(x) = – \sin x – (xf(x) + \int^{x}_{0} f(t)dt) + xf(x)$

$= –\sin x – \int^{x}_{0} f(t)dt$ $\quad$ $f''(x) = – \cos x – f(x)$

Illustration : *If a function f(x) is defined ∀x ∈ R such that . Prove that*

$\int^{a}_{x} f(t)/t\, dx$. Prove that $\int^{a}_{0} g(x)\, dx = \int^{a}_{0} f(x)dx$

Solution: $g(x) = \int^{a}_{x} f(t)/t\, dt$

Diffrentiate w.r.t. x

$g'(x) = – F(x)/x$

$F(x) = -x\, g'(x)$

$\int^{a}_{0} f(x)dx = –\int^{a}_{0} xg'(x)\, dx$

$= \int^{a}_{0} f(x)dx = –ag(a) + \int^{a}_{0} g(x)\, dx = \int^{a}_{0} g(x)dx,$ $\quad$ [g(a) = 0]

Illustration **:** Determine a positive integer n £ 5, such that

$\int_{10} e^{x} (x–1)^{n} dx = 16 – 6e$

Solution: Let $I_n = \int^{1}_{0} ex\, (x–1)^{n} dx$

Integrating by parts,

$I_n = I_n = [e^{x} (x–1)_{n}]^{1}_{0} – \int^{1}_{0} e^{x}.n(x–1)dx = (–1)^{n+1} = nI_{n–1}$ $\quad$ ...(i)

Also, $I_1 = \int^{1}_{0} e^{x}.n(x–1)^{n–1} dx = [(x–1)e^{x}]^{1}_{0} – \int^{1}_{0} e^{x}ex.1dx$

$= –(–1)–[e^{x}]^{1}_{0} = 1–(e–1) = 2 – e$

From (i), $I_2 = (-1)^3 - 2I_1 = -1 - 2(2 - e) = -5 + 2e$

and $I_3 = (-1)^4 - 3I_2 = 1 - 3(-5 + 2e) = 16 - 6e$ Which is given .

$\therefore n = 3$

2.5 Definite Integral As Limit Of A Sum

An alternative way of describing $\int_a^b f(x)dx$ is that the definite integral $\int_a^b f(x)\,dx$ is a limiting case of the summation of an infinite series, provided f(x) is continuous on [a, b] i.e., $\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh)$ where h = b–a/n. The converse is also true i.e., if we have an infinite series of the above form, it can be expressed as a definite integral.

Method to express the infinite series as definite integral:

(i) Express the given series in the form $\sum 1/n\, f(r/n)$

(ii) Then the limit is its sum when $n \to \infty$, i.e. $\lim_{n\to\infty} h \sum 1/n f(r/n)$

(iii) Replace r/n by x and 1/nby dx and $\lim_{n\to\infty} \sum$ by the sign of $\int$.

(iv) The lower and the upper limit of integration are the limiting values of r/n for the first and the last term of r respectively.

Some particular cases of the above are

(a) $\lim_{n\to\infty} \sum_{r=1}^{n} 1/n\, f(r/n)$ or $\lim_{n\to\infty} \sum_{r=0}^{n-1} 1/n\, f(r/n) = \int_0^1 f(x)dx$

(b) $\lim_{n\to\infty} \sum_{r=1}^{pn} 1/n\, f(r/n) = \int_\alpha^\beta f(x)dx$

where $\alpha = \lim_{n\to\infty} r/n = 0$ (as r = 1) and $\beta = \lim_{n\to\infty} r/n = p$ (as r = pn)

Illustration *:* *Show that*

(A) $\lim_{n\to\infty} (1/n+1 + 1/n+2 + 1/n+3 + ... + ... 1/n+n) = \ln 2.$

(B) $\lim_{n\to\infty} 1^p + 2^p + 3^p + ... + / n^{p+1} = 1/p+1$ (p > 0)

Solution: (A) Let $I = \lim_{n\to\infty} (1/n+1 + 1/n+2 + 1/n+3 + ... + 1/n+n)$

$= \lim_{n\to\infty} (1 / n+1 + 1/n+2 + 1/n+3 + ... + 1/n+n)$

$= \lim_{n\to\infty} 1/n \sum_{r=1}^{n} 1 / 1+r/n$

Now $\alpha = \lim_{n\to\infty} 1/n = 0$ (as r = 1)

and $\beta = \lim_{n\to\infty} r/n = 1$ (as r = n)

$=> I \int_0^1 1/1+x\, dx = [\ln (1+x)]_0^1 => I = \ln 2.$

(B) $1^p + 2^p + 3^p + ... + n^p / n^{p+1} = \sum^n_{r=1} 1^p/n.n^p = \sum^n_{r=1} 1+n(r/n)^p$

Take $f(x) = x^p$; Let $h = 1/n$ so that as $n \to \infty$; $h \to 0$

$\therefore \quad \lim_{n\to\infty} \sum^n_{r=1} 1/n\, f(0 + r/n)$

$= \int_0^1 f(x)dx = \int_0^1 x^p dx$

$= 1/p+1$

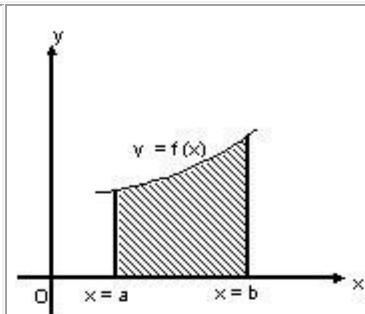
2.6 Section c – AREA AS DEFINITE INTEGRAL

Let f (x) be a continuous function in (a, b). Then the area bounded by the curve y = f (x),

x axis and lines x = a and x = b is given by the formula

$A = |\int_a^b f(x)dx|$,

provided $f(x) \geq 0$ (or $f(x) \leq 0$) $\forall x \in (a, b)$

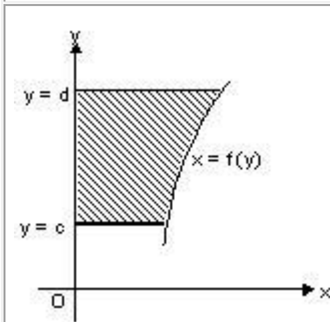


It is sometimes convenient to use formula for area with respect to y i.e. regarding x as a function of y.

The area between x = f(y), y axis and the lines

y = c and y = d is given by

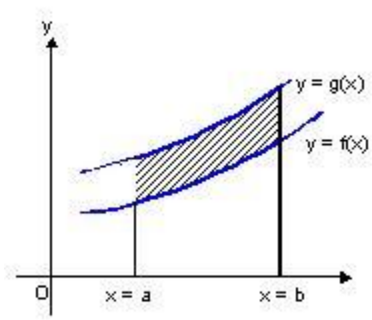
$A = \int_c^d f(y)dy$



If we have two functions f(x) and g(x) such that $f(x) \leq g(x)$ $\forall x \in [a, b]$, then the area bounded by the curves y = f(x), y = g(x) and lines x = a,

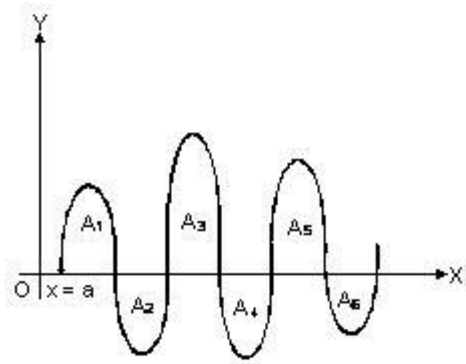
x = b (a < b) is given by

$A = \int_a^d [g(x)-f(x)]dx$

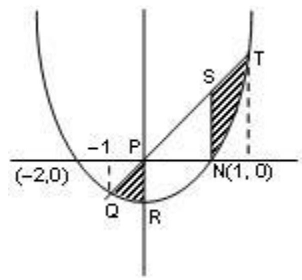


2.7 **Working Rule for finding the Area :**

(i) If curve lies completely above the x axis, then the area is positive but when it lies completely below x axis, then the area is negative, however we have the convention to consider the magnitude only.



(ii) If curve lies on both the sides of x axis i.e. above the x axis as well as below the x axis, then calculate both areas separately and add their modulus to get the total area.



In general if curve y = f(x) crosses the x axis n times when x varies from a to b, then the area between y = f(x), x axis and lines x = a and x = b is given by

$$A = |A_1| + |A_2| + ... + |A_{11}|$$

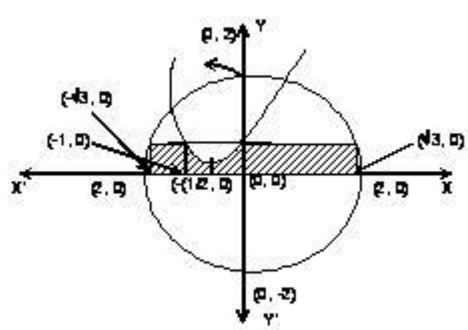
(iii) If the curve is symmetrical about x axis, or y axis, or both, then calculate the area of one symmetrical part and multiply it by the number of symmetrical parts to get the whole area.

***Illustration** : Find the area between the curves $y = x^2 + x - 2$ and $y = 2x$, for which $|x^2 + x - 2| + |2x| = |x^2 + 3x - 2|$ is satisfied.*

Solution: $y = x^2 + x - 2 \Rightarrow y = 2x$

$|x^2 + x - 2| + |2x| = |x^2 + 3x - 2|$

$(x^2 + x - 2)$ and $2x$ have same sign



Thus required area = ar (PQR) + ar (ECD)

$$= \int_{-1}^{0} [2x - (x^2 + x - 2)]\, dx + \int_{1}^{2} [2x - (x^2 + x - 2)]\, dx$$

$= [x^2/2 - x^3/3 + 2x]^0_{-1} + [x^2/2 - x^3/3 + 2x]^2_1$
$= 7/6 + [10/3 - 13/6] = 7/6 + 7/6 = 7/3$

Illustration *: Find out the area enclosed by circle $|z| = 2$, parabola $y = x^2 + x + 1$, the curve $y = [\sin^2 x/4 + \cos x/4]$ and x-axis (when [.] is the greatest integer function).*

Solution: For $x \in [-2, 2]$

$\Rightarrow 1 < \sin^2 x/4 + \cos x/4 < 2$

$\therefore [\sin^2 x/4 + \cos x/4] = 1$

Now we have to find out the area enclosed by the circle $|z| = 2$,

parabola $(y - 3/4) = (x+1/2)^2$,

line y = 1and x-axis.

∴ Required area is shaded area in the figure.

Hence required area $= \sqrt{3} \times 1 + (\sqrt{3} - 1) + \int^0_1 (x^2 + x + 1).dx + 2 \int^2_{\sqrt{3}} \sqrt{4-x^2}\, dx$

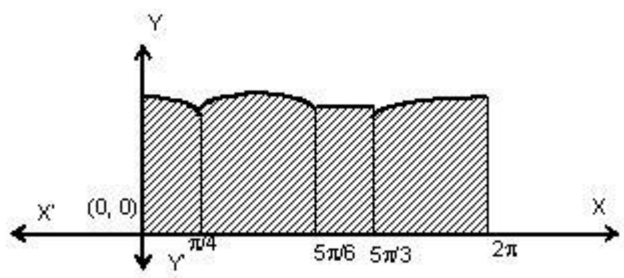
$= (2\pi/3 + \sqrt{3} - 1/6)$ sq. units

Illustration : *Let f(x) = Max. {sinx, cos x, 1/2} then determine the area of the region bounded by the curves y = f(x), x-axis, y-axis and x = 2π.*

Solution: f(x) = Max {sin x, cos x, 1/2}

interval value of f(x)

for $0 \leq x < \pi/4$, cos x



for $\pi/4 \leq x < 5\pi/6$, sin x

for $5\pi/6 \leq x < 5\pi/3$, 1/2

for $5\pi/3 \leq x \leq 2\pi$ cos x

Hence required area

$= \int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{5\pi/6} \sin x\,dx + \int_{5\pi/6}^{5\pi/3} 1/2.dx + \int_{5\pi/3}^{2\pi} \cos x\,dx$

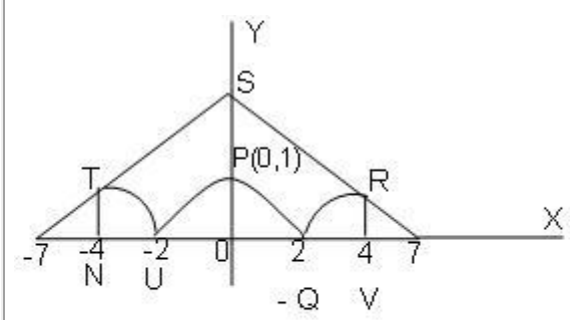
$= (5\pi/12 + \sqrt{2} + \sqrt{3})$ sq units.

Illustration : *Find area bounded by* $y = |4 - x^2|/4$ *and* $y = 7 - |x|$

Solution: $y = |4 - x^2|/4$ (I)

$y = 7 - |x|$ (II)

Rewriting (I) and (II)

$y = \{4-x^2 / 4 \quad -2 \le x \le 2$ $x^2-4/4 \quad x \in (-\infty, -2) \cup (2, \infty)$ $y = \{7 - x \quad x \ge 0$ $7 + x \quad x < 0$ Required area (PQRSTUP) = 2 area (PQRSP) = 2 [area of (OVRSO) – area of (POQP) – area of (QRVQ)]	

$= 2 [\ 1/2\,(7 + 3)4 - \int_0^2 (4-x^2/4)\,dx - \int_2^4 x^2-4/4\,dx]$

$= 2\ 18 + 2/3 - 16/3 + 4 + 2/3 - 2$

= 32 sq unit.

Illustration *:* *Let f be a real valued function satisfying f(x/y) and* $\lim_{x\to 0} f(1+x)/x = 3$*. Find the area bounded by the curve y = f(x), the y axis and the line y = 3*

Solution: Given = f(x/y) = f(x) – f(y) .... (1)

Putting x = y = 1, we get f(1) = 0

Now, $f'(x) = \lim_{h\to 0} f(x+h) - f(x) / h$

$= \lim_{h\to 0} f(1+h/x)/h$ (from(1))

$= \text{Lim}_{h\to 0} f(1+h/x)\,(h/x)x$

$\Rightarrow \quad f'(x) = 3/x$ $\quad\quad\quad\quad$ {since $\text{Lim}_{x\to 0} f(1+x0/x = 3$}

$\Rightarrow \quad f(x) = 3 \ln x + c$

Putting x = 1

$\Rightarrow \quad c = 0$

$\Rightarrow \quad f(x) = 3 \ln x = y$ (say)

Required area $= \int^{3}_{-\infty} x\, dy = \int^{3}_{-\infty} e^{y/3}\, dy = 3[e^{y/3}]^{3}_{-\infty}$

$= 3\,(e - 0) = 3e$ sq. units

Illustration : *Let A_n be the area bounded by $y = \tan^n x$, $x = 0$, $y = 0$ and $x = p/4$. Prove that for n ³ 2.* $\quad$ *(i) $A_n + A_{n-2} = 1/n-1$* $\quad$ *(ii) $1/2(n+1) < A_n < 1/2(n-1)$*

Solution: (i) Obviously $A_n = .\int^{\pi/4}_{0} \tan^n x dx$

$A_n + A_{n-2} = \int^{\pi/4}_{0} (\tan^n x dx + \tan^{n-2} x)dx$

$= \int^{1}_{0} t^{n-2}\, dt = 1/n-1.$

(ii) Obviously $A_{n+2} < A_n < A_{n-2}$ (from figure).

Thus $2A_n = A_n + A_n < A_n + A_{n-2} = 1/n-1$ (by part (i)

Thus $A_n < 1/2(n-1)$ $\quad$ . . . . (1)

Also $2A_n = A_n + A_n > A_n + A_{n+2} = 1/n+1$, replacing n by n+2 in (i)

$A_n > 1/2(n+1)$ $\quad$ . . . . (2)

From (1) and (2), we get $1/2(n+1) < A_n < 1/(2n-1)$.

1. OBJECTIVE PROBLEMS

1: $\displaystyle\int_{0}^{200\pi} \sqrt{\frac{1-\cos 2x}{2}}\,dx$

(A) ***0*** $\quad$ ***(B)*** ***100***

(C) ***200*** $\quad$ ***(D)*** ***400***

Solution :

$$I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c$$

$= -200\ (\cos x)^{\pi}_{0} = -200\ (-1-1) = 400$

2: *$\int^{2}_{-2}$ min (x–[x], – x – [–x])dx, where [.] is the greatest integer function =*

(A) ***0*** ***(B)*** ***1***

(C) ***2*** ***(D)*** ***None of these.***

Solution : Let I = $\int^{2}_{-2}$ min (x – [x], – x – x [–x])dx

= 2 $\int^{2}_{-2}$ min (x – [x], – x – [–x])dx (function is even)

$$\textbf{min}\,(x-[x], -x-[-x]) = \begin{cases} x, & 0 \le x < \frac{1}{2} \\ -x+1, & \frac{1}{2} \le x < 1 \\ x-1, & 1 \le x < \frac{3}{2} \\ -x+2, & \frac{3}{2} \le x < 2 \end{cases}$$

$$\therefore I = 2\int_0^{\frac{1}{2}} x\,dx + 2\int_{\frac{1}{2}}^{1}(-x+1)dx + 2\int_{1}^{\frac{3}{2}}(x-1)dx + 2\int_{\frac{3}{2}}^{2}(-x+2)dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + 2\left[-\frac{x^2}{2}+x\right]_{\frac{1}{2}}^{1} + 2\left[\frac{x^2}{2}-x\right]_{1}^{\frac{3}{2}} + 2\left[-\frac{x^2}{2}+2x\right]_{\frac{3}{2}}^{2}$$

= 1/4 + 1/4 + 1/4 + 1/4 = 1

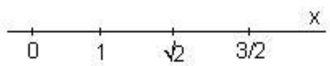
Hence I = 1

3: ***(where [.] denotes greatest integer function) =***

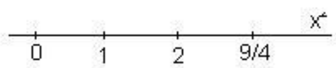
(A) 2-√2 ***(B)*** 2+√2

(C) 2√2 ***(D)*** √2

Solution : $\int_0^{3/2}[x^2]\,dx$



$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{3/2} 2\,dx$



= (√2 – 1) + 2 (3/2 – √2) = √2 – 1 + 3 – 2 √2 = 2 + √2 – 2 √2 = 2 – √2

4: ***$Q^{p/2}{}_0$ sin 2 x in tan x dx =***

(A) 2 ***(B)*** 0

(C) 1 ***(D)*** None of these.

Solution : I = $\int_0^{\pi/2}$ sin 2x ln (tan x) dx … (1)

I = $\int_0^{\pi/2}$ sin 2 (π/2 – x) ln tan (π/2 – x)dx

I = $\int_0^{\pi/2}$ sin 2x ln (cot x) dx … (2)

Adding (1) and (2)

2I = $\int_0^{\pi/2}$ sin 2x [ln (tan x) + ln (cot x)] dx

= $\int_0^{\pi/2}$ sin 2x ln 1 dx = 0.

5: ***$O^2{}_{1/2}$ 1/x sin e/e x – 1/x o/o dx =***

(A) 0 ***(B)*** 1

(C) 2 ***(D)*** *√2*

Solution : Let I = $\int_{1/2}^{2}\frac{1}{x}\frac{\left(1+\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)}\sin\left(x-\frac{1}{x}\right)dx$

Put z = x - 1/x

Then I = $\int_{-3/2}^{3/2}\frac{\sin z}{\sqrt{z^2+4}}dz = 0$ as $\frac{\sin z}{\sqrt{4+z^2}}$ is odd function

6: ***The value of*** $\int_0^{\pi/2}$ √sin x / √sin x + √cos x is

(A) ***p/2*** ***(B)*** ***p/3***

(C) п/***4*** ***(D)*** ***p***

Solution : Let I = $\int_0^{\pi/2}$ √sin x / √sin x + √cos x dx

Here the lower limit is zero hence , we can replace x by (a – x) i.e. by p/2 –x

$$\therefore I = \int_0^{\pi/2}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx = \int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$\text{Adding } 2I = \int_0^{\pi/2}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

7: $\lim_{n\to\infty}\left[\tan\frac{\pi}{3n}+\tan\frac{2\pi}{3n}+....+\tan\frac{\pi}{3}\right]\frac{1}{n}$ =

(A) ***3/p log 2*** ***(B)*** ***2/p log 2***

(C) ***3/p log 3*** ***(D)*** ***3/p***

Solution : $\lim_{n\to\infty}\frac{1}{n}\left(\sum_{r=1}^{n}\tan\left(\frac{r\pi}{3n}\right)\right)$

r/n = x, 1/n = dx

$$\int_0^{\pi/3} \tan\left(\frac{x\pi}{3}\right) dx = \frac{3}{\pi}\left|\ln(\sec t)\right|_0^{\pi/3} = \frac{3}{\pi}\ln 2 \qquad (\because t = \pi x/3)$$

8: The area bounded by the curve $y = x(3 - x)^2$, the x – axis and the ordinates of the maximum and minimum points of the curve is

(A) 2 sq. units ***(B)*** 6 sq. units

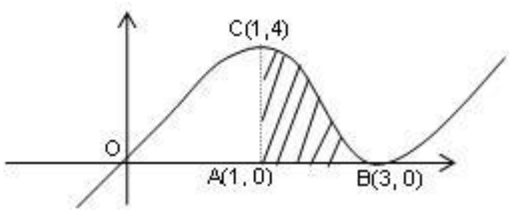
(C) 4 sq. units ***(D)*** 8 sq. units

Solution : $y = x(3 - x)^2$

After solving ,
we get x =1 and x = 3 are points of maximum and minimum respectively.

Now the shaded region is the required region

$$\therefore \ A = \int_1^3 x(3 - x)^2\, dx = 4 \text{ sq. units}$$



9: What is the area of a plane figure bounded by the points of the lines max (x, y) = 1 and $x^2 + y^2 = 1$?

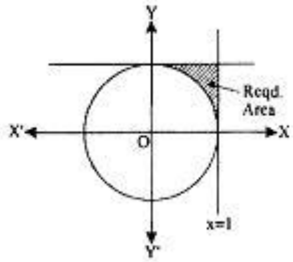
(A) ***p/2*** sq. units ***(B)*** p/3 sq. units

(C) ***p/4*** sq. units ***(D)*** ***p*** sq. units

Solution : By definition the lines max, (x, y) = 1 means.

$x = 1$ and $y \leq 1$ or $y = 1$ and $x \leq 1$

Required area



$$= \int_0^1 \left[1 - \sqrt{1 - x^2}\right] dx$$

$$= \left[x - \frac{x}{2}\sqrt{1 - x^2} - \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 1 = 0 = \frac{1}{2}\left(\frac{\pi}{2}\right) = 1 = \frac{\pi}{4}\,sq\,units$$

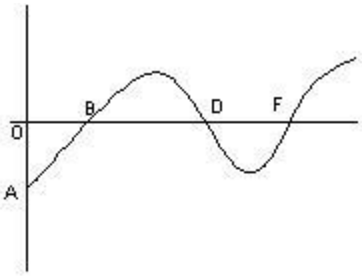
10: The area bounded by the curve y = (x –1) (x – 2) (x – 3) lying between the ordinates x = 0 and x = 3 is

(A)	*7/4* sq. units	***(B)***	4 sq. units
(C)	*11/ 4* sq. units	***(D)***	3 sq. units

Solution : Reqd. area $= \int_0^3 |y|\,dx = \int_0^1 |y|\,dx + \int_1^2 |y|\,dx + \int_2^3 |y|\,dx$

$$= -\left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)_0^1 + \left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)_1^2 - \left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)_2^3$$

= 9 / 4 + 1/ 4 + 1/ 4 = 11/4 sq units.



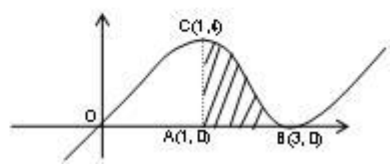
Solved Examples of Definite Integral Part I

11: The area bounded by the curve $y = x(3 - x)^2$, the x – axis and the ordinates of the maximum and minimum points of the curve is

(A) 2 sq. units **(B) 4 sq. units**

(C) 3 sq. units **(D) 8 sq. units**

Solution: $y = x(3 - x)^2$



After solving , we get x =1 and x = 3 are points of maximum and minimum respectively.

Now the shaded region is the required region

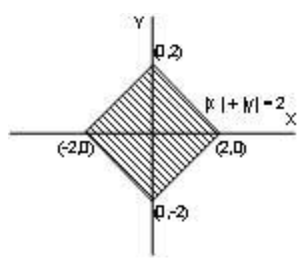
$$\therefore\ A = \int_1^3 x(3-x)^2\,dx = 4 \text{ sq. units}$$

12: The area enclosed by the curve |x + 1| + |y + 1| = 2 is

(A) 3 sq. units **(B) 4 sq. units**

(C) 5 sq. units **(D) 8 sq. units**

Solution: Shift the origin at the point (-1, -1)



So |x + 1| + |y + 1| = 2

becomes |x| + | y | = 2

Hence required area is

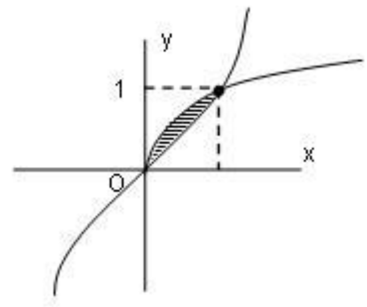
= 4 × 1/2 × 2 × 2 = 8sq. units

13: The area common to the curves y = x^3 and y = $\sqrt{x}$ is

(A) 2 **(B) 4**

(C) 8 **(D) None of these**

Solution: $A = \int_0^1 (\sqrt{x} - x^3)\,dx$



$= \frac{2}{3}\left|x^{3/2}\right|_0^1 - \left|\frac{x^4}{4}\right|_0^1$

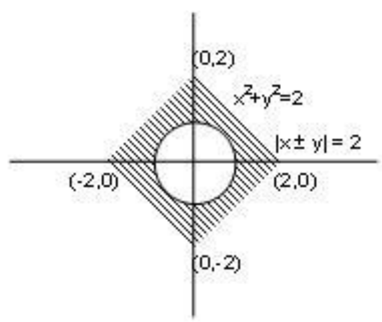
$= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$

14: The area of the region consisting of points (x, y) satisfying

$|x \pm y| \leq 2$ and $x^2 + y^2 \geq 2$ is

(A) 8 – 2π sq. units **(B) 4 – 2π sq. units**

(C) 1 – 2π sq. units **(D) 2π sq. units**



Solution: Shaded region is the required one.

Required Area = 4 × 1/2 × 2 × 2 – π.2 = 8 – 2π sq. unit

15: $\int_o^x$ [sint] dt where x ∈ and [2nπ, (4n+1)]π, n ∈ N and [.] denotes the greatest integer function is equal to.

(A) **-nπ** **(B)** **-(n +1)π**

(C) **-2nπ** **(D)** **-2(n +1)π**

Solution: $I = \int_0^x [\sin t]dt = \int_0^{2n\pi} [\sin t]dt + \int_{2n\pi}^{x} [\sin t]dt$

$$= n\int_0^{2\pi} [\sin t]dt + \int_{2n\pi}^{x} [\sin t]dt$$

$$= n\left(\int_0^{\pi/2} [\sin t]dt + \int_{\pi/2}^{\pi} [\sin t]dt + \int_{\pi}^{2\pi}[\sin t]dt\right) + \int_{2n\pi}^{x} [\sin t]dt$$

$$\Rightarrow I = n\left(0 + 0 - \int_{\pi}^{2\pi} dt\right) + 0 = -n\pi$$

16: If f(π) = 2 and $\int_0^{\pi}$ [f(x) + f''(x)] sin x dx = 5 then f(0) is equal to, (it given that f(x) is continuous in [0, π])

(A) **7** **(B)** **3**

(C) **5** **(D)** **1**

Solution: $\int_0^{\pi} (f(x) + f''(x)) \sin x \, dx$

$$= \int_0^{\pi} f(x).\sin x \, dx + \int_0^{\pi} f''(x) \sin x \, dx$$

$$= f(x).-\cos x\Big|_0^{\pi} + \int_0^{\pi} \cos x.f'(x)dx + \sin x \,.\, f'(x)\Big|_0^{\pi} - \int_0^{\pi} \cos x.f'(x)dx$$

=> f(**π**) + f(0) = 5 (given)

=> f(0) = 5 – f(**π**) = 5 – 2 = 3

17: Let f(x) is a continuous function for all real values of x and satisfies

$$\int_0^x f(t)dt = \int_x^1 t^2 . f(t)dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$$

then value of 'a' is equal to

(A) −1/24 (B) 17/168

(C) 1/7 (D) None of these

Solution: $\int_0^x f(t)dt = \int_x^1 t^2 . f(t)dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$.........(i)

$$\text{For } x = 1,\ \int_0^1 f(t)dt = 0 + \frac{1}{8} + \frac{1}{3} + a = \frac{11}{24} + a$$

Diff. both sides of (i) w. r. t. x we get;

$f(x) = 0 - x^2 f(x) + 2x^{15} + 2x^5$

$$\Rightarrow 2\int_0^1 \frac{x^{15} + x^5}{1 + x^2} dx = \frac{11}{24} + a$$

$$\Rightarrow 2\int_0^1 \left(x^{13} - x^{11} + x^9 - x^7 + x^5\right)dx = \frac{11}{24} + a$$

$$\Rightarrow 2\int_0^1 \left(\frac{1}{14} - \frac{1}{12} + \frac{1}{10} - \frac{1}{8} + \frac{1}{6}\right) = \frac{11}{24} + a \Rightarrow a = -\frac{167}{840}$$

18: $f(x) = \int_1^x \frac{e^t}{t} dt,\ x \in R^+$**. Then complete set of values of x for which f(x) $\le$ lnx is**

(A) (0, 1] (B) [1, ∞)

(C) (0, ∞) (D) None of these

Solution: $f(x) = \int_0^x \frac{e^t}{t} dt \Rightarrow f(1) = 0 \text{ and } f'(x) = \frac{e^x}{x}$

Let g(x) = f(x) - ln(x). $x \in R^+$

=> $g'(x) = f'(x) - 1/x = e^x - 1/x > 0 \ \forall x \in R^+$

=> g'(x) is increasing for $\forall x \in R^+$

g(1) = f(1) - ln1 = 0 – 0 = 0

=> g(x) > 0 ∀ x > 1 and $g(x) \geq 0$ $\forall x \in (0, 1]$

=> $\ln x \geq f(x) \ \forall x \in (0, 1]$

19: $\int_0^x \frac{2^t}{2^{[t]}} dt$ **, where [.] denotes the greatest integer function, and $x \in R^+$, is equal to**

(A) **$\frac{1}{\ln 2}([x] + 2^{[x]}-1)$** **(B)** **$\frac{1}{\ln 2}([x] + 2^{[x]})$**

(C) **$\frac{1}{\ln 2}([x] - 2^{[x]})$** **(D)** **$\frac{1}{\ln 2}([x] + 2^{[x]}+1)$**

Solution: Let $n \leq x < n + 1$ where $n \in I, I \geq 0$

$$I = \int_0^x \frac{2^t}{2^{[t]}} dt = \int_0^n 2^{\{t\}} dt + \int_n^x 2^{\{t\}} dt$$

$$= n\int_0^1 2^{\{t\}} dt + \int_n^x 2^{\{t\}} dt = n\int_0^1 2^t dt + \int_0^x 2^{t-n} dt$$

$$= n.\frac{2^t}{\ln 2}\Big|_0^1 + \frac{1}{2^n}.\frac{2^t}{\ln 2}\Big|_n^x$$

$$= n.\frac{1}{\ln 2}(2-1) + \frac{1}{2^n.\ln 2}(2^x - 2^n) = \frac{[x] + 2^{\{x\}} - 1}{\ln 2}$$

20: ***Let I_1 =*** $\int_0^{\pi/4} e^{x^2} dx$, $I_2 = \int_0^{\pi/4} e^x dx$, $I_3 = \int_0^{\pi/4} e^{x^2}.\cos x \, dx$, $I_4 = \int_0^{\pi/4} e^{x^2}.\sin x \, dx$ ***then***

(A) ***$I_1 > I_2 > I_3 > I_4$*** ***(B)*** ***$I_2 > I_3 > I_4 > I_1$***

(C) ***$I_3 > I_4 > I_1 > I_2$*** ***(D)*** ***$I_2 > I_1 > I_3 > I_4$***

Solution: **(b)** $x > x^2 \; \forall x \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow e^x > e^{x^2} \; \forall x \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow$Since $\cos x > \sin x \; \forall \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow e^{x^2} . \cos x > e^{x^2} \sin x$

$\Rightarrow \quad e^x > e^{x^2} > e^{x^2} . \cos x > e^{x^2} \sin x \; \forall x \in \left(0, \frac{\pi}{4}\right) \quad \Rightarrow \quad I_2 > I_1 > I_3 > I_4$

Solved Examples

30. ***The value of $\int_0^{100} (\sqrt{x}) dx$ (where {x} is the fractional part of x) is***

(A) 50 ***(B) 1***

(C) 100 ***(D) none of these***

Solution: Given integral = $\int_0^{100} (\sqrt{x} - [\sqrt{x}]) dx$ (by the def. of {x})

$$= \int_0^{100} \sqrt{x}\,dx - \sum_{i=1}^{10} \int_{(i-1)^2}^{i^2} [\sqrt{x}]dx$$

$$= \left(\frac{2}{3}x^{3/2}\right)_0^{100} - 2\sum_{i=1}^{10} \int_{i-1}^{i} [t]dt \quad \text{where } t^2 = x$$

$$= \frac{2000}{3} - 2\left(\int_0^1 0.dx + \int_1^2 1.dx + \int_2^3 2.dx + ... + \int_9^{10} 9.dx\right)$$

$$= \frac{2000}{3} - 2\left([x]_1^2 + [2x]_2^3 + [3x]_3^4 + ... + [9x]_9^{10}\right)$$

$$= \frac{2000}{3} - 2\left(\frac{(9)(9+1)}{2}\right) = \frac{1730}{3}.$$

Hence (D) is the correct answer.

31. ***The value of $\int_0^1$ (|sin 2 p x| dx is equal to***

(A) 0 ***(B) 2/π***

(C) 1/π ***(D) 2***

Solution: Since |sin 2 π x | is periodic with period 1/2,

$I = \int_0^1 |\sin 2\pi x|\,dx = 2\int_0^1 \sin 2\pi x\,dx$

$= 2\,[-\cos 2\pi x/2\pi]_0^{1/2} = 2/\pi$

Hence (B) is the correct answer.

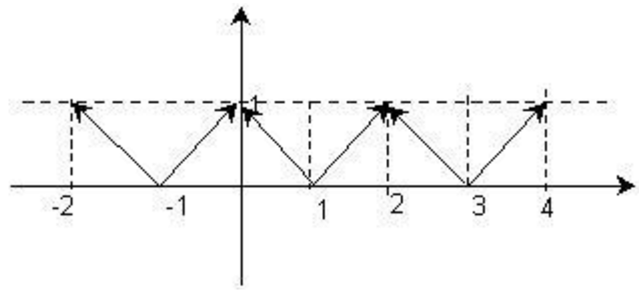
32. ***Let f : R —> R, f(x) =*** $\begin{cases} |x - [x]|, & [x] \text{ is odd} \\ |x - [x+1]|, & [x] \text{ is even} \end{cases}$***, where [.] denotes greatest integer function, then $\int_{-2}^{4} f(z)dz$ is equal to***

(A) 5/2 ***(B) 3/2***

(C) 5 ***(D) 3***

Solution: $x - [x]$
$= \{x\}$

$x - [x+1] = \{x\} - 1$



$\int_{-2}^{4} f(x)dx$

$= 6.1/2\ (1.1) = 3$

Hence (D) is the correct answer.

33. $\int_{-\pi/4}^{\pi/4} \frac{e^x (x \sin x)}{e^{2x} - 1} dx$ ***is equal to***

(A) 0 ***(B) 2***

(C) e ***(D) none of these***

Solution: $I = \int_{-\pi/4}^{\pi/4} \frac{e^x (x \sin x)}{e^{2x} - 1} dx$

property $\int_{-a}^{a} f(x)dx = 0$ ($f(-x) = -f(x)$, odd function)

Hence I = 0

Hence (A) is the correct answer.

34. The value of $\int_{-10}^{10} 3^x/3^{[x]}\, dx$ is equal to (where [.] denotes greatest integer function) :

(A) 20 (B) 40 / In3

(C) 20 / In 3 (D) none of these

Solution: $I = \int_{-10}^{10} 3^{x-[x]} dx = 20\int_0^1 3^{x-[x]} dx = 20\int_0^1 3^x dx$

$$= 20\left|\frac{3^x}{\ln 3}\right|_0^1 = 20\ \frac{1}{\ln 3}\ (3-1) = \frac{40}{\ln 3}.$$

Hence (B) is the correct answer.

35. Values of $\int_{-1/2}^{+1/2} \cos x \log 1+x/1-x \, dx$ is :

(A) 1/2 (B) – 1/2

(C) 0 (D) none of these

Solution: $I = \int_{-1/2}^{+1/2} \cos x \log 1+x/1-x \, dx$

$f(x) = \cos x \ln 1+x/1-x$

$f(-x) = \text{cox}(-x) \ln 1+x/1-x$

$= -\cos(x) \ln(1+x/1-x) = -f(x)$

f (x) is an odd function

hence I = 0

Hence (C) is the correct answer.

36. $f(x) = \min(\tan x, \cot x)$, $0 \leq x \leq$, then $\int_0^{\pi/2} f(x)dx$ is equal to :

(A) ln2 (B) ln √2

(C) 2 ln √2 (D) none of these

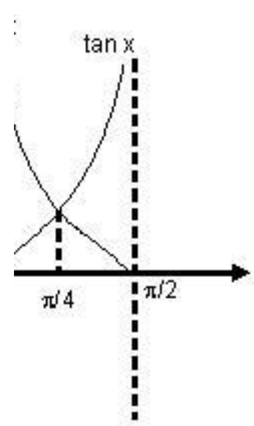
Solution: $f(x) = \min(\tan x, \cot x)$,

$\in [0, \pi/2]$

$f(x) = \tan x, \quad 0 \leq x \leq \pi/4$

$= \cot x, \quad \pi/4 < x \leq \pi/4$

Hence



$$= \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/2} \cot x\,dx$$

$$I = |\ln(\sec x)|_0^{\pi/4} + |\ln(\sin x)|_{\pi/4}^{\pi/2}$$

$$= (\ln\sqrt{2} - 0) + \left(0 - \ln\frac{1}{\sqrt{2}}\right)$$

$2 \ln \sqrt{2} = \ln 2.$

Hence (A) is the correct answer.

37. The value of $|\cos x|_{-\pi/2}^{0} - |\cos x|_{0}^{\pi/2}$ is equal to :

(A) π/2 (B) 2π

(C) π (D) π/p

Solution: I = $\int_{-1}^{3}\left[\tan^{-1}\left(\frac{x}{x^2+1}\right) + \cot^{-1}\left(\frac{x}{x^2+1}\right)\right]dx = \int_{-1}^{3}\frac{\pi}{2}\,dx = 2\pi$

Hence (B) is the correct answer.

38. The value of $\int_0^1 \left(1 + e^{-x}\right)dx$ is equal to :

(A) 2 – 1/e (B) 2 + 1/e

(C) e+1/e (D) none of these

Solution: I = $\int_0^1 \left(1 + e^{-x}\right)dx$ $= |x - e^{-x}|_0^1 \; (1 - e^{-1}) - (0 - 1) = 2 - e^{-1}$

Hence (A) is the correct answer.

39. $\int_{-1}^{1} \{[x] + |x|\}dx$ has the value is :

(A) 0 (B) 1/2

(C) 1 (D) 1/4

Solution: $\int_{-1}^{1}([x]+|x|)\,dx = \int_{-1}^{0}(-1-x)\,dx + \int_{0}^{1}(0+x)\,dx$

$= \left|-x-\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2}\right|_{0}^{1} = 0-\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-0\right) = 0$

Hence (A) is the correct answer.

40. $\int_{0}^{\pi/2}\frac{dx}{1+\tan^3 x}$ is :

(A) 0 (B) 1

(C) π/2 (D) π/4

Solution: $I = \int_{0}^{\pi/2}\frac{\cos^3 x}{\sin^3 x+\cos^3 x}\,dx$

$I = \int_{0}^{\pi/2}\frac{\sin^3 x}{\sin^3 x+\cos^3 x}\,dx \Rightarrow 2I = \int_{0}^{\pi/2}dx \Rightarrow I = \frac{\pi}{4}$

Hence (D) is the correct answer.

41. The value of $\int_{-\pi/2}^{\pi/2}\left(p\sin^3 x+q\sin^4 x+r\sin^5 x\right)dx$ depends on :

(A) p (B) q

(C) r (D) p and q

Solution: I = $\int_{-\pi/2}^{\pi/2}(p\sin^3 x+q\sin^4 x+r\sin^5 x)\,dx$

= q $\int_{-\pi/2}^{\pi/2}\sin^4 x$ (Since $\sin^3x$ and $\sin^5 x$ are odd functions)

Hence (B) is the correct answer.

42. The value of $\int_{-\pi/2}^{\pi/2} \sin|x|\,dx$ is equal to :

(A) 2 (B) −2

(C) 1 (D) 0

Solution: I = $\int_{-\pi/2}^{0} -\sin x\,dx + \int_{0}^{\pi/2} \sin x\,dx$

$|\cos x|_{-\pi/2}^{0} - |\cos x|_{0}^{\pi/2}$ = (1 − 0) − (0 − 1) = 2.

Hence (A) is the correct answer.

43. Value of $\int_0^n [x]\,dx$ (where n ∈ N) is

(A) n(n+1)/2 (B) n(n−1)/2

(C) n (n − 1) (D) None of these.

Sol: $\int_0^n [x]\,dx = \sum_{i=1}^{n} \int_{i-1}^{i} [x]\,dx$

$\sum_{i=1}^{n} (i-1)\,dx = \frac{n(n-1)}{2}$

Hence (B) is the correct answer.

44. $\int_{-8}^{8} (\sin^{193} x + x^{295})\,dx$ is equal to

(A) 0 (B) 2 (829^5 + 1)

(C) 829^5 + 2 (D) None of these.

Sol: $\sin^{193} x + x^{295}$ is an odd function of x.

? The given integral is zero.

Hence (A) is the correct answer.

45. If $F(x) = 2\int_0^x f(t)\,dt$ and $f(x) = \begin{cases} x, & 0 \le x < 1 \\ \sin \pi x, & x \ge 1 \end{cases}$, then

(A) $F'(1/2) = 1/4$ (B) $F'(3/2) = -1$

(C) $F'(1) = 1$ (D) None of these.

Sol: .$F'(x) = f(x) \Rightarrow F'(3/2) = \sin 3\pi/2 = -1$

Hence (B) is the correct answer.

46. The value of $\int_{-2}^{2} (ax^3 + bx + c)\,dx$ depends on

(A) b (B) c

(C) a (D) a and b.

Sol: $I = \int_{-2}^{2} (ax^3 + bx)\,dx + c\int_{-2}^{2} dx = 0 + c \cdot 4 = 4c$

(is an odd function of x).

Hence (B) is the correct answer.

47. $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx$ (m ≠ n and m, n are integers) =

(A) 0 (B) π

(C) π/2 (D) 2π

Sol: $I = \frac{1}{2}\int_0^{\pi} 2\sin mx \cos nx \, dx$

$= \frac{1}{2}\int_0^{\pi} \cos(m-n)x - \cos(m+n)x \, dx$

$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi} = 0$

Hence (A) is the correct answer.

48. $\int_0^{\infty}\left(a^{-x} - b^{-x}\right) dx =$

(A) 1/loga – 1/logb (B) log a – log b

(C) log a + log b (D) 1/loga + 1/logb

Sol: $= \int (a^{-x} - b^{-x})dx = \left[\frac{a^{-x}}{-\log a} - \frac{b^{-x}}{-\log b}\right]_0^{\infty} = 0 + \frac{a^0}{\log a} - \frac{b^0}{\log b} = \frac{1}{\log a} - \frac{1}{\log b}$

Hence (A) is the correct answer.

49. $\int_{-1}^{1} \sin^{11} x \, dx =$

(A) 10 × 8 × 6 × 4 × 2 / 11 × 9 × 7 × 5 × 3 (B) 10 × 8 × 6 × 4 × 2 / 11 × 9 × 7 × 5 × 3 π/2

(C) 1 (D) 0

Sol: $\sin^{11} x$ is odd function of x.

So, integral is zero.

Hence (D) is the correct answer.

50. If $I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$, then $I_8 + I_6$

(A) 1/4 (B) 1/5

(C) 1/6 (D) 1/7

Sol: $I_8 + I_6 = \int_0^{\pi/4} (\tan^8\theta + \tan^6\theta)\, d\theta$

$= \int_0^{\pi/4} \tan^6\theta \sec^2\theta\, d\theta = \left[\frac{\tan^7\theta}{7}\right]_0^{\pi/4} = \frac{1}{7}$

Hence (D) is the correct answer.

51. $\int_0^{\pi/2} \sin^7 x\, dx =$

(A) 37/184 (B) 17/45

(C) 16/35 (D) 16/45

Sol: .U = 6 × 4 × 2 / 7 × 5 × 3 × 1 = 16/35

Hence (C) is the correct answer.

52. $\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} =$

(A) π/ab (B) π/2ab

(C) πab (D) $\pi^2 ab$

Sol: Divide Num. and Den. by $\cos^2 x$ and put tan x = t etc.

Hence (B) is the correct answer.

53. If g (x) = $\int_0^x \cos^4 t\, dt$, then g (x + π) =

(A) g(x) + g(π) (B) g(x) − g(π)

(C) g(x) g(π) (D) g(x)/g(π)

Sol: $g(x+\pi) = \int_0^{\pi+x} \cos^4 t\,dt$

$$= \int_0^{\pi} \cos^4 t\,dt + \int_{\pi}^{\pi+x} \cos^4 x\,dx$$

(Put $t = \pi + \theta$ is second integral)

$$.= \int_0^{\pi} \cos^4 t\,dt + \int_0^{x} \cos^4 t\,dt = g(\pi) + g(x)$$

Hence (A) is the correct answer