Rotational Motion

Rigid Body:

A rigid body is one for which the distance between any pair of points on the object remains fixed.
If an object changes its shape or size, then the distance between some pairs of point will change. Hence a rigid body retains its shape and size under the application of forces. The concept of rigid body is an idealization.

Translation and Rotation of a Rigid Body:

In translational motion, all the particles in a rigid object have the same displacement in the same time interval. In rotational motion, all the particles in a rigid object execute circular motion about the axis of rotation.

Static Equilibrium of a Rigid Body :

An extended object is in static equilibrium if every point of that object remains at rest .A rigid body is an object for which the distance between any pair of points on the object remains fixed.

A rigid body in static equilibrium neither translates nor rotates and is , therefore, in translational and rotational equilibrium.

Translational Equilibrium:

As we know that the motion of the centre of mass is determined by the external forces,

$$M\, a_{cm} = \Sigma F_{ext}$$

Where M is mass of the object.

The object is said to be translational equilibrium if the acceleration of centre of mass zero. Mathematically; If $a_{cm}=0$, then $\sum F_{ext}=0$ This is the condition for translational equilibrium.

Rotational Equilibrium :

For rotational equilibrium about any point $\sum\tau_{ext} = 0$ must be satisfied. For coplanar forces , these conditions reduce to

$$\Sigma F_{x,ext} = 0, \qquad \Sigma F_{y,ext} = 0, \qquad \Sigma\tau_{z,ext} = 0$$

Where the coplanar forces lie in the x-y plane. These equations can be solved for up to three unknowns.

Centre of Gravity:

The centre of gravity of an extended object is that point at which the full gravitational force on the object can be considered to act. The centre of gravity and the centre of mass coincide for the objects of ordinary size close to earth surface.

Centre of Mass :

The centre of mass of a body is a point where the entire mass of the body can be supposed to be concentrated. Infact, nature of motion executed by the body shall remain unaffected if all the forces acting on the body were applied directly at this point.

For a system of two particles of masses m_1 and m_2 having their position vectors as $\vec{r}_1$ and $\vec{r}_2$ respectively, with respect to origin of the coordinate system, the position vector of the centre of mass is given by

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

If $m_1 = m_2 = m$(say), then $\vec{R}_{CM} = \frac{r_1 + r_2}{r}$

Thus, the centre of mass of two equal masses lies exactly at the centre of the line joining the two masses.

For a system of N-particles of masses m_1, m_2, m_3… m_4 having their position vectors as $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_N$ respectively, with respect to the origin of the coordinate system, the position vector of the centre of mass is given by

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_N\vec{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{M}$$

The coordinates of centre of mass are given by

$$X_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

$$Y_{CM} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i y_i}{M}$$

$$Z_{CM} = \frac{\sum_{i=1}^{N} m_i z_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

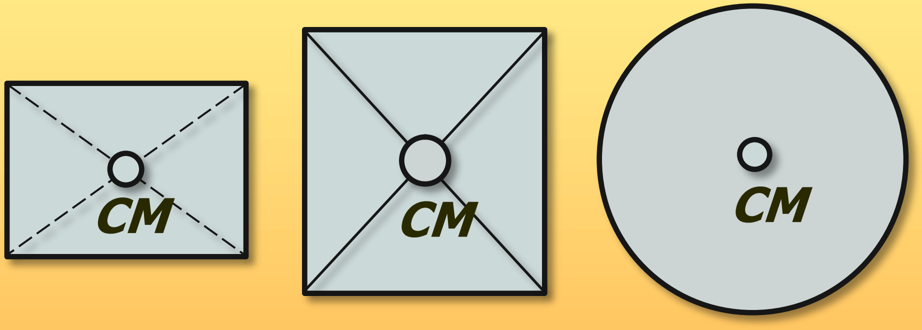
For a continuous distribution of mass, the coordinates of centre of mass are given by

$$X_{CM} = \frac{1}{M}\int x\,dm; \; Y_{CM} = \frac{1}{M}\int y\,dm; \; Z_{CM} = \frac{1}{M}\int z\,dm;$$

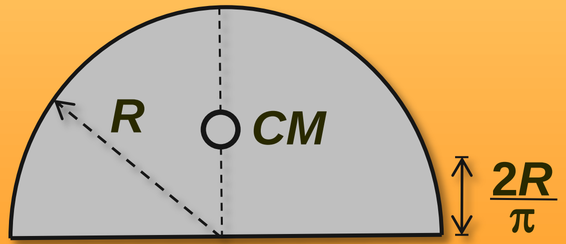
- **The position of the centre of mass of a system is independent of the choice of coordinate system.**
- **The position of the centre of mass depends on the shape of the body and the distribution of its mass. Hence it may lie within or outside the material of the body.**
- **In symmetrical bodies in which the distribution of mass is homogeneous, the centre of mass coincides with the centre of symmetry of geometrical centre.**
- **The centre of mass changes its position only under the translatory motion but remains unchanged in rotatory motion.**

Centre of mass of some well known rigid bodies are given below :

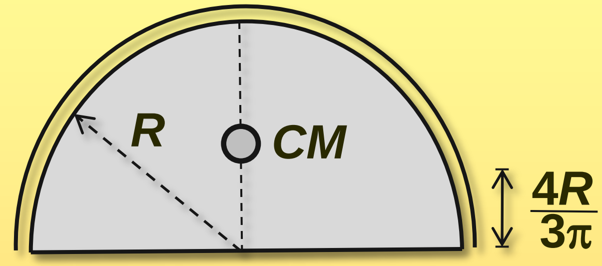
- **Centre of mass of a uniform rectangular, square or circular plate lies at its centre as shown in the figure.**



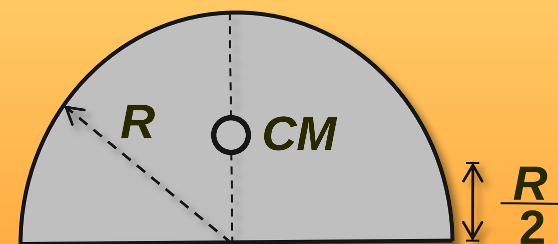
- **Centre of mass of a uniform semicircular ring of radius *R* lies at a distance of $h = \frac{2R}{\pi}$ from its centre, on the axis of symmetry as shown in the figure.**



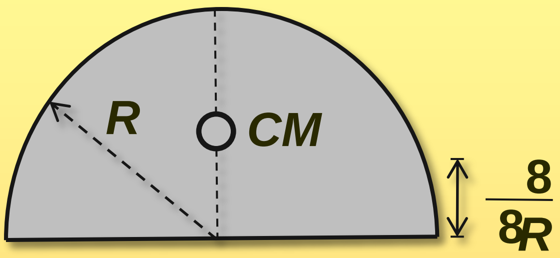
- Centre of mass of a uniform semicircular disc of radius R lies at a distance of $h = \frac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in figure.



- Centre of mass hemispherical shell of radius R lies at a distance of $h = \frac{R}{2}$ from its centre on the axis of symmetry as shown in figure.



- **Centre of mass of a solid hemisphere of radius R lies at a distance of $h = \frac{3R}{8}$ from its centre on the axis of symmetry as shown in the figure**



Velocity of centre of mass is given by

$$\vec{v}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_N \vec{v}_N}{m_1 + m_2 + \ldots + m_N}$$

$$= \frac{\sum_{i=1}^{N} m_i \vec{v}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \vec{v}_i}{M}$$

Acceleration of centre of mass is given by

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_N\vec{a}_N}{m_1 + m_2 + \ldots + m_N}$$

$$= \frac{\sum_{i=1}^{N} m_i\vec{a}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\vec{a}_i}{M}$$

If total external force acting on the system is zero, then the total linear momentum of the system is conserved. Also , when the total external force acting on the system is zero, the velocity of centre of mass remains constant.

Angular measurement:

The SI unit angle, the rod, is defined as $\theta = s/R$, where s and R are measured with the same length unit.

Angular coordinate, velocity, and acceleration:

Angular kinematical quantities are the angular coordinate θ,
the angular velocity component $\omega_z = d\theta/dt$,
the angular acceleration component $\alpha_z = d\omega_z/dt$.
Note : The right-hand rule is used to define the positive sense for rotation.

Equations for uniformly accelerated angular motion are exactly analogous to those for uniformly accelerated linear motion. In usual notation we have :

Linear Motion	Rotational Motion
$v = v_0 + at$	$\omega = \omega 0 + \alpha t$
$s = v_0 t + \frac{1}{2} a t^2$	$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$
$v^2 - v_0^2 = 2ax$	$\omega^2 - \omega_0^2 = 2\alpha\theta$

Kinematics of rotation about a fixed axis:

The kinematics of a rigid object rotating about a fixed axis is analogous to that of a particle moving in a straight line.

Relations between angular and linear velocity and angular and linear acceleration:

For a particle in a rotating rigid object, the relation between the linear and angular velocity components are

$$V = R\omega$$

The linear acceleration components are:

- **Transverse component of linear acceleration** $a_t = R\alpha_z$
- **Radial component of linear acceleration** $a_R = R\omega^2$ **(directed towards the centre)**

Rotational kinetic energy: moment of inertia:

The rotational kinetic energy of a rigid object is

$$K = \frac{1}{2} I \omega^2$$

Where the moment of inertia is

$$I = \sum m_i R_i^{\ 2}$$

Moment of inertia:

The moment of inertia of a continuous object is

$$I = \int_V p R^2 dV$$

Theorem of perpendicular axis : The moment of inertia of a planar lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with

perpendicular axis and lying in the plane of the body.

$$I_z = I_x + I_y$$

where x and y are two perpendicular axes in the plane and z axis is perpendicular to its plane.

Theorem of parallel axes : The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

$$I = I_{cm} + Md^2$$

where I_{cm} is the moment of inertia of the body about an axis passing through the centre of mass and d is the perpendicular distance between two parallel axes.

Note : Use the definition of the moment of Inertia to develop expressions for moments of inertia; use the parallel-axis theorem.

Rolling objects:

The axis of rotation of a rolling object uncle goes translation.

The kinetic energy of a rolling object can be written

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v^2$$

Note : Use rotational and translational quantities to describe the motion of a rolling object.

Angular momentum of a particle:

The angular momentum $\mathbf{l}$ of a particle about a point O is

$$\mathbf{l} = r \times p$$

where r is measured relative to O and p = mv. From New-ton's second law, the relation between $\mathbf{l}$ and the net torque

$$\sum \tau = r \times \sum F$$

$$\sum \tau = \frac{d\mathbf{l}}{dt}$$

Note : Determine the angular momentum of a particle traveling in a circle and of a particle traveling in a straight line;

Determine the rate of change of the angular momentum of a particle from the net torque exerted on it.

Angular momentum of a system of particles:

The total angular momentum of a system of particles is the vector sum of the angular moment of the particles that compose :

$$L = \sum l_i$$

and

$$\sum \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt}$$

where $\sum \tau_{ext}$ is the net external torque on the system.

Note:

Determine the total angular momentum of a system of particles; use Newton's laws to show that internal torques do not contribute to the rate of change of the total angular momentum.

Rotational dynamics of a rigid object about a fixed axis:

For the case of a rigid object rotating about a fixed axis,

$$L_z = I\omega_z \quad \text{and} \quad \sum \tau_z = I\alpha_z$$

Relative to any point on the axis, a torque's axial component is

$$\tau_z = RF_t$$

Note:

Use the rotational analog of Newton's second law to determine the rotational motion a rigid object rotating about a fixed axis.

Rotational work and power for a rigid object:

The work done by a torque on a rigid object rotating about a fixed axis is

$$W = \int_{\theta i}^{\theta_f} \tau_z d\theta$$

Work done by the net torque changes the rotational kinetic energy:

$$I_{net} = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$

The power to a rotating rigid object is

$$P = \tau_z \omega_z$$

Conservation of angular momentum :

If the net external torque on a system is zero, the total angular momentum of the system is conserved:

$$L_i = L_f$$

Note : Use the principle of conservation of angular momentum to determine rotational motion.

The following table summarizes the mathematical similarity between translational motion in one dimension and rotational motion about a fixed axis.

Translation (one dimension)	Rotation (fixed axis)
$\sum \mathrm{F}_{\mathrm{ext},x} = Ma_x$	$\sum \tau_{\mathbf{ext,z}} = \mathbf{I}\boldsymbol{\alpha}_{\mathrm{z}}$
$\mathrm{P}_x = \mathrm{Mv}_x$	$\mathrm{L}_{\mathbf{z}} = \mathbf{I}\boldsymbol{\omega}_{\mathrm{z}}$
$\mathrm{W} = \int F_x dx$	$\mathrm{W} = \int \tau_z \mathrm{d}\theta$
$W_{\mathrm{net}} = \frac{1}{2} M\mathrm{v}_{f^2} - \frac{1}{2} M\mathrm{v}_{i^2}$	$\mathrm{W}_{\mathrm{net}} = \frac{1}{2} I\omega_{f^2} - \frac{1}{2} I\omega_{i^2}$
$P = F_x \mathrm{v}_x$	$P = \tau_z \omega_z$

Note : **The torque component τ_z is positive if it tends to produce a counterclockwise rotation of the object when viewed from the positive z axis, and τ_z is negative if the tendency of rotation is clockwise.**

Description of symbols used in above table:

For rotational motion		For translational motion	
Coordinate	θ	Coordinate	X
Velocity component	ω_z	Velocity component	v_x
Acceleration component	α_z	Acceleration component	a_x
Moment of inertia	I	Mass	M
Torque component	τ_z	Force component	F_x
Momentum	L_z	Momentum	P_x

Moment of inertia of different bodies:

Shape of body	Axis of rotation	Moment of inertia
1. Thin rod	Perpendicular to length	$M l^2 / 12$
2. Disc	Perpendicular to plane of disc	$M r^2 / 2$
3. Ring	Perpendicular to plane of ring	$M r^2$
4. Cylinder	Axis of cylinder	$M r^2 / 2$
5. Cylinder	Perpendicular to axis of cylinder	$M\left[\frac{l^2}{12} + \frac{r^2}{4}\right]$
6. Sphere	Diameter	$2 M r^2 / 5$
7. Spherical shell	Diameter	$2 M r^2 / 3$

Steps For Solving a Simple Static Problem

- Sketch the situation showing the rigid body which is in static equilibrium.
- Construct the free-body diagram by drawing in all external force vectors acting on the rigid body and indicate the magnitude, the direction, and the point of application of each force. Some of these quantities will be unknown; for example, you may not know the direction of a force or the point at which it is applied.
- Select a set of coordinate axes along which to resolve the forces. Usually, a judicious choice of the orientation of the coordinate axes simplifies this resolution.

- Make a choice of an axis about which to evaluate torques. Choosing the axis to pass through the intersection of the lines of action of two or more forces is often convenient because the moment arm of each of these forces is zero.

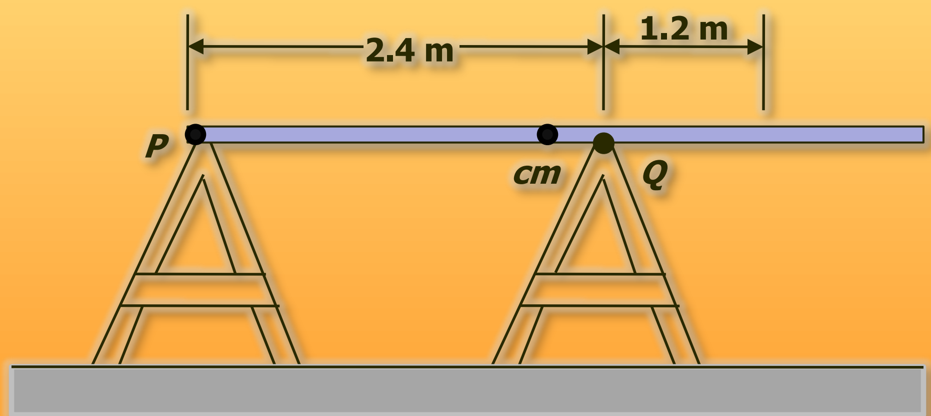
- Apply the conditions of static equilibrium,.

- Solve these equations for up to three unknowns.

One of the forces which acts on a rigid body close to the earth's surface is the weight of the body. Its point of application in the examples to follow is assumed to be at the center of mass of the object. We shall discuss this assumption in the next section.

Example 1.

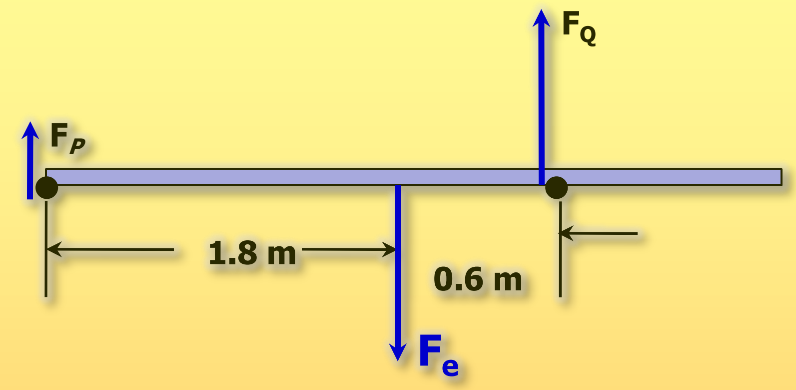
A uniform 48-N board of length 3.6m rests horizontally on two sawhorses, as shown in below figure (a).What normal forces are exerted on the board by the sawhorses ?



(a)

Solution:

Let us draw the free-body diagram of the board.



(b)

Since the board is uniform, the weight is assumed to act at the center of mass, 1.8 m from each end of Figure (b) shows the free-body diagram for the board. The sawhorses exert vertical normal forces of magnitude F_p and F_Q on the board at points P and Q. These two normal forces are the unknowns.

Since the board is in static equilibrium, we can apply the conditions of static equilibrium, .

Take the x axis as horizontal and they axis as vertically upward.

Then $\sum F_{x,ext} = 0$ is automatically satisfied because all the forces act vertically.

Requiring $\sum F_{y,ext} = 0$,
we have

$$F_p + F_Q - F_e = 0,$$

or

$$F_p + F_Q - F_e$$

The sum of the two normal forces must balance the weight. We must now select an axis about which to calculate torques. Any axis will do, but a convenient choice is the axis through point P because F_p exerts no torque, and F_Q exerts a counterclockwise torque about the axis through P.
With the z axis perpendicularly out of the slide,

applying $\sum\tau_{z,ext} = 0$
Gives

$$(2.4\ \text{m})\, F_Q - (1.8\ \text{m})\, F_e = 0,$$

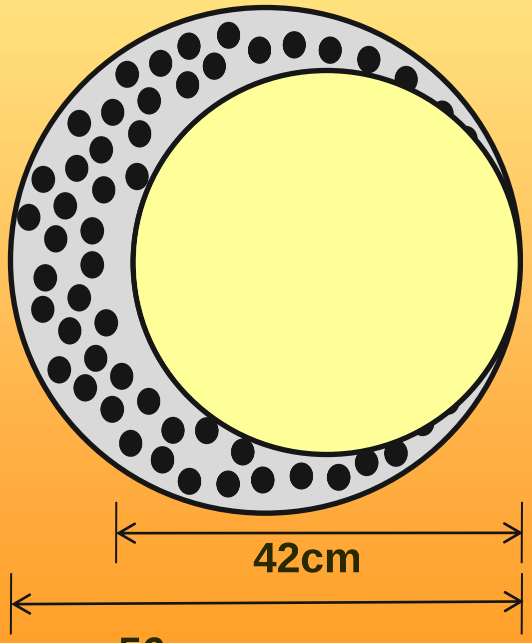
or $$(2.4\text{m})F_Q = (1.8\ \text{m})\,(48\ \text{N})$$

$$F_Q = 36\ \text{N}$$

The other normal force can be determined immediately since $F_P + F_Q = 48$ N,
or $F_P = 12$ N. (By putting the value of)

Note: **Try reworking this example for a different choice of axis, such as through point O. or through the center of mass.**

- **Example 1 :** A circular plate of uniform thickness has a diameter of 56cm. A circular portion of diameter 42cm is removed from one edge of the plate as shown in the figure. Find the position of the centre of mass of the remaining portion.



Solution : Let distance of centre of mass from centre of bigger circle = r_1

$\therefore \quad A_1 r_1 = A_2 r_2$

where A = area = πR^2

or $\quad r_1 = \left(\dfrac{A_2}{A_1}\right) r_2$

or $$r_1 = \frac{\pi R_2^2}{\pi\left(R_1^2 - R_2^2\right)}\left(R_1 - R_2\right) = \frac{R_2^2\left(R_1 - R_2\right)}{\left(R_1 + R_2\right)\left(R_1 - R_2\right)}$$

$$r_1 = \frac{R_2^2}{\left(R_1 + R_2\right)}$$

or $$r_1 = \frac{4R_2^2}{4\left(R_1 + R_2\right)} = \frac{\left(42\right)^2}{2\left(56 + 42\right)} = \frac{42 \times 42}{2 \times 98} = 9\text{cm}$$

Since the portion is cut off from right, centre of mass shifts to left.

$\therefore$ **Centre of mass of remaining portion is at 9cm to left, from centre of bigger circle.**

- **Example 2 :** **A physics professor is seated on a stool rotating about a vertical axis with an angular speed ω_i. The professor's arms are outstretched, and she is holding a dumbbell in each hand such that the moment of inertia of the system (professor, stool seat, and dumbbells) is I_i . She quickly pulls the dumbbells in to her sides so that the final moment of inertia of the system is one-third the initial: $I_f = I_i/3$**

  (a) What is her final angular speed?
  (b) Compare the final and initial kinetic energies of the system. Neglect the torque due to friction in the stool's axle during the time interval in which the system's moment of inertia changes.

Solution : (a) Since the torque due to friction in the stool's axle is negligible, there are no external torques on the system and its angular momentum is conserved :

$$I_i\omega_i = I_f\omega_f = \frac{I_i}{3}\omega_f$$

Solving for ω_f, we have

$$\omega_f = 3\omega_i$$

Thus conservation of angular momentum requires that the angular speed increase by the same factor by which the moment of inertia decreases.

(b) The final kinetic energy of the system is

$$K_f = \tfrac{1}{2} I_f \omega_f{}^2 = \frac{1}{2}\left(\frac{I_i}{3}\right)\left(3\omega_i\right)^2$$

$$= 3\left(\tfrac{1}{2} I_f \omega_f^{\,2}\right) = 3K_i$$

Thus conservation of angular momentum requires that the kinetic energy of the system increase by the same factor by which the moment of inertia decreases.

Think About It!

Where did this additional kinetic energy come?

Thanks…