MATHEMATICAL INDUCTION AND BINOMIAL THEOREM

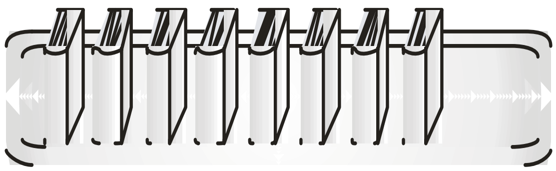
Contents

3. MATHEMATICAL INDUCTION AND BINOMIAL THEOREM

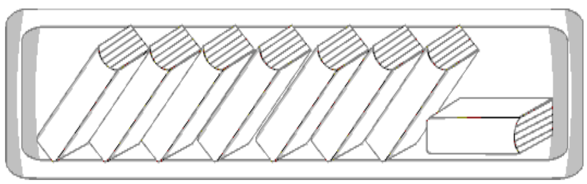
3.1 Mathematical Induction

3.1.1 The principle of Mathematical Induction

The word 'Induction' means method of reasoning from individual cases to general ones or from observed instances to unobserved ones. Many important mathematical formulae are such that a result is formed by some means which does not provide for a direct proof. Mathematical Induction is a principle by which one can arrive at a conclusion about a statement for all positive integers, after proving certain related proposition. Let us say there are books lined up as shown in the figure.



We know if we push over one book, the rest the books fall over as seen in the figure below.



a. If we push over one book, it should fall.
b. If a book is falling and has been placed correctly, it will knock over its neighbour. Though intuitively we can say that if the next to the last book falls, the last book falls, but this needs to be proved by logical reasoning.

Now let us consider the following:

1. Assume that there is some book 'k' which doesn't fall over i.e., k is the first book which behaves in a different manner.
2. Since k is the first book, the book right before k must have fallen over.
3. But we know from B, a falling book always knocks over the next one.
4. So book k will fall over and we have a contradiction.

We think of each book as an instance of a proposition. If a given instance is true, the corresponding book will fall over, given a sequence of instances (row of books).
If we can prove

1. The proposition is true in the first instance.
2. And if a given instance is true.
3. The next one in the sequence will also be true.

Then the proposition will be true in all instances.

This is called proving a proposition by Induction.

THE NATURAL NUMBERS are the counting numbers: 1, 2, 3, 4, etc. Mathematical induction is a technique for proving a statement - a theorem, or a formula that is asserted about *every* natural number.

By "every" natural number, or "all" natural numbers, we mean any one that we might possibly name.

For example,

$$1 + 2 + 3 + \ldots + n = \frac{1}{2} n (n + 1).$$

This asserts that the sum of consecutive numbers from 1 to *n* is given by the formula on the right. We want to prove that this will be true for *n*=1, *n* = 2, *n* = 3, and so on. Now we can test the formula for any *given* number, say *n* = 3:

$1 + 2 + 3 = \frac{1}{2} \cdot 3 \cdot 4 = 6$ ……………… which is true .

It is also true for n = 4:

$1 + 2 + 3 + 4 = \frac{1}{2} \cdot 4 \cdot 5 = 10$

How to prove this rule for every value of n?

The method of proof is shown below. This is called the principle of mathematical induction.

3.1.1.1 First principle of mathematical induction (weak mathematical induction):

The set of statements {P(n): n $\in$ N }is true for each natural number n $\geq$ m provide that:

1. P(m) is true.
2. P (k) is true for n=k, (where k $\geq$ m) $\Rightarrow$ P (n) is true for n =k+1

Working Rule

Let there be a proposition or a mathematical statement namely P(n), involving a natural number n. In order to prove that P(n) is true for all natural numbers n $\geq$ m, we proceed as follows:

1. Verify that P(m) is true.
2. Assume that P(k) is true (where k $\geq$ m)
3. Prove that P(k+1) is true.

Once step-3 is completed after 1 and 2, we can conclude that P(n) is true for all natural numbers n $\geq$ m.

3.1.1.2 Second Principle of Mathematical Induction (Strong mathematical induction):

The set of statements {P(n): n $\in$ N}is true for each natural number n $\geq$ m provided that:

1. P(m) and P(m+1) are true.
2. P(n) is true for n $\leq$ k (where k $\geq$ m) $\Rightarrow$ P(n) is true for n=k+1

This is also called **extended principle of Mathematical Induction.**

Working Rule

1. Verify that P(n) is true for n=m, n=m+1.
2. Assume that P(n) is true for n $\leq$ k (where k $\geq$ m).
3. Prove that P(n) is true for n=k+1.

Once step-3 is completed after 1 and 2, we can conclude that P(n) is true for all natural numbers n $\geq$ m.

(This method is to be used when P(n) can be expressed as a combination of P(n-1) and P(n-2). In case P(n) turns out to be a combination of P(n-1), P(n-2), and P(n-3), we verify for n=m+2 also in step 1)

3.1.2 Solved Examples

We proceed to illustrate the use of the above principles by means of a few examples.

- **Problems based on divisibility:**

Example: Show that $10^{2n-1}+1$ is divisible by 11 for all natural numbers n.

Solution: Let $P(n)=10^{2n-1}+1$

$P(1)=10^1+1=11$ which is clearly divisible by 11.

Let $P(k)=10^{2k-1}+1$ be divisible by 11.

$P(k+1)= 10^{2k+1}+1$

$= 10^{2k-1}.10^2+1$

$=[10^{2k-1}+1]+99.\ 10^{2k-1}$

$=P(k)+99.\ 10^{2k-1}$ which is divisible by 11.

Hence P(k+1) is divisible by 11.

Hence, by mathematical induction, the result is true for all n.

Example: Show that $11^{n+2}+12^{2n+1}$ is divisible by 133 for every natural number n.

Solution: Let $P(n)= 11^{n+2}+12^{2n+1}$

$P(1) =11^3+12^3=3059$ which is divisible by 133.

Let $P(k)=11^{k+2}+12^{2k+1}$ be divisible by 133.

$P(k+1)=11^{k+3}+12^{2k+3}$

$=11^{k+2}.11+12^{2k+1}\ 144,$

$=11.11^{k+2}+(133+11)12^{2k+1}$

$=11[11^{k+2}+12^{2k+1}]+133.12^{2k+1}$

$=11.P(k)+133.12^{2k+1}$

P(k) is divisible by 133 and so is 133.12^{2k+1}.

Hence, P(k+1) is also divisible by 133.

Hence, by mathematical induction, the result is true for all n.

Example: Show by induction $x(x^{n-1}-na^{n-1})+a^n(n-1)$ is divisible by $(x-a)^2$ for all positive integers (n > 1).

Solution: $P(n)=x(x^{n-1}-na^{n-1})+a^n(n-1)$

$P(2)=x(x-2a)+a^2(2-1)$

$=x^2-2ax+a^2=(x-a)^2$

$\therefore$ P(2) is true.

Let P (k) be divisible by $(x-a)^2$ for some positive integers k>1, then

$P(k)=x(x^{k-1}-ka^{k-1})+a^k(k-1)$

$=(x-a)^2\ P(x,a)$

$\Rightarrow x^k=kxa^{k-1}-a^k(k-1)+(x-a)^2\ P(x, a)$

Thus $P(k+1)=x[x^{k+1-1}-(k+1)a^{k+1-1})+a^{k+1}(k+1-1)$

$=x^k.x-(k+1)xa^k+ka^{k+1}$

$=[kxa^{k-1}-a^k(k-1)+(x-a)^2P(x, a)]x-(k+1)xa^k+ka^{k+1}$

$=kx^2a^{k-1}-2kxa^k+ka^{k+1}+x(x-a)^2 P(x,a)$

$=ka^{k-1}(x^2-2xa+a^2)+x(x-a)^2 P(x,a)$

$=(x-a)^2 [ka^{k-1}+xP(x, a)]$

Hence P(k+1) is divisible by $(x-a)^2$.

- **Problems based on summation of series:**

Example: Use mathematical induction to show that $1+3+5+... +(2n-1)=n^2$ is true for all natural numbers n.

Solution: Let $P(n)=1+3+5+...+(2n-1)=n^2$

P (1) $=1=1^2$, which is true.

Assume that P(k) holds good.

$\Rightarrow$ $P(k)=1+3+5+...+(2k-1)=k^2$

$P(k+1)=[1+3+5+...+(2k-1)]+(2k+1)$

$=P(k)+2k+1=k^2+2k+1=(k+1)^2$

Hence, P(k+1) is true.

Hence, by mathematical induction, the result is true for all $n\in$ N.

Example: Using mathematical induction, show that

$$\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{3^2}\right]\ldots..\left[1-\frac{1}{(n+1)^2}\right]=\frac{n+2}{2n+2}, n\in N$$

Solution: Let $P(n)=\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{3^2}\right]\ldots..\left[1-\frac{1}{(n+1)^2}\right]=\frac{n+2}{2n+2}$

L.H.S of P(1)= $\left[1-\frac{1}{2^2}\right]=\frac{3}{4}$ =R.H.S

Hence P(1) is true.

Assume that P(k) is true.

$$\Rightarrow\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{3^2}\right]\ldots..\left[1-\frac{1}{(k+1)^2}\right]=\frac{k+2}{2k+2}$$

For P(k+1), the L.H.S becomes

$$\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{3^2}\right]\ldots..\left[1-\frac{1}{(k+1)^2}\right]\left[1-\frac{1}{(k+2)^2}\right]$$

$$=P(k)\left[1-\frac{1}{(k+2)^2}\right]=\frac{k+2}{(2k+2)}\left[\frac{k^2+4k+3}{(k+1)^2}\right]$$

$$=\frac{k+2}{2k+2}\frac{(k+1)(k+3)}{(k+2)^2}=\frac{(k+2)(k+1)(k+3)}{2\,(k+1)(k+2)^2}$$

$$=\frac{k+3}{2(k+1)+2}\Rightarrow P\,(k+1) \text{is true}$$

Hence, by mathematical induction, the result is true for all $n\in$ N

Example: Using mathematical induction, show that $\sum_{r=0}^{n} r.^{n}C_r = n.2^{n-1}$

Solution: Let P(n)= $1.^{n}C_1 + 2.^{n}C_2 + \cdots + n.^{n}C_n = n\,2^{n-1}$

P(1) $= 1.^{1}C_1 = 1$(L.H.S)

$= 1.\ 2^{1-1} = 1$ (R.H.S)

$\Rightarrow$ P(1) is true.

Assume that P(k) is true $\Rightarrow 1.^{k}C_1 + 2.^{k}C_2 + \cdots + k.^{k}C_k = k\,2^{k-1}$

To prove that P(k+1) is true, we write

$\sum_{r=0}^{k+1} r.\ ^{k+1}C_r = 1.^{k+1}C_1 + 2.^{k+1}C_2 + \cdots + (k+1).^{k+1}C_{k+1}$

$\sum_{r=0}^{k} r.\ ^{k+1}C_r + (k+1) = \sum_{r=0}^{k} r\,[^{k}C_r + \ ^{k}C_{r-1}] + (k+1)$

$= P(k) + \sum_{r=0}^{k} r.^{k}C_r + k + 1.^{k}C_k$

Changing r-1 to r, we get

$P(k+1) = P(k) + \sum_{r=0}^{k} r + 1.^{k}C_r + k + 1.^{k}C_k$

$= k.2^{k-1} + \sum_{r=0}^{k}(r+1)^{k}C_r = k.2^{k-1} + \sum_{r=0}^{k} r.^{k}C_r + \sum_{r=0}^{k} k_{C_r}$

$= k\,2^{k-1} + P(k) + 2^k = 2k\,2^{k-1} + 2^k = 2^k(k+1)$

Hence the result is true for P(k+1).

Hence, by mathematical induction, the result is true for all n.

- **Problems involving equations:**

Use of transitive property:

Suppose it is given F(n)> G(n) or $\frac{F(n)}{G(n)} > 1$ we have to prove that

$F(n+1) > G(n+1)$ or $\frac{F(n+1)}{G\,(n+1)} > 1$

or $\frac{F(n+1)}{G(n+1)} > \frac{F(n)}{G(n)} > 1$ *or* $\frac{F(n+1)}{F(n)}\frac{G(n)}{G(n+1)} > 1$

Example: Show that $^{2n}C_n < 4^n \quad \forall\, n \in N$

Solution: Show $^{2n}C_n < 4^n$

F(n)= 4^n, G(n) = $^{2n}C_n$

F(n+1)= 4^{n+1}, G(n+1) = $^{2n+2}C_{n+1}$

$$\frac{F(n+1)}{F(n)}.\frac{G(n)}{G(n+1)} = \frac{4^{n+1}}{4^n}.\frac{^{2n}C_n}{(\ ^{2n+2}C_n)}$$

$$= 4.\frac{(2n)!}{n!\,n!}.\frac{(n+1)!\,(n+1)!}{(2n+2)!} = 2.\frac{(n+1)}{(2n+1)} > (\text{Since } 2n+2 > 2n+1)$$

$\Rightarrow \frac{F(n+1)}{G\,(n+1)}.\frac{G(n)}{F(n)} > 1$

or $\frac{F(n+1)}{G\,(n+1)} > \frac{F(n)}{G(n)} > 1 \Rightarrow$ F(n+1)>G(n+1)

Example: Using mathematical induction show that $\tan \alpha > n \tan \alpha$ where $0 < \alpha < \frac{\pi}{4(n-1)}$ $\forall$ natural numbers, n>1.

Solution: Since n>1 we start with n=2.

$\Rightarrow \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} > 2\tan\alpha, \text{since } 1-\tan^2\alpha < 1.$

Hence, the result holds for n=2

Suppose it holds for n=k

$\Rightarrow \tan k\alpha > k \tan\alpha.$

For n=k+1,

$\tan(k+1)\alpha = \frac{\tan k\,\alpha + \tan\alpha}{1-\tan k\,\alpha\tan\alpha} > \frac{k\,\tan\,\alpha + \tan\alpha}{1-\tan k\,\alpha\tan\alpha} > (k+1)\tan\alpha, \text{since } 1-\tan k\,\alpha\tan\alpha < 1.$

Hence the result holds for n=k+1.

Hence, by mathematical induction, the result is true for all n.

Example: Show, using mathematical induction, that

$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1} > 1$ For all natural numbers n.

Solution: Let us test for n=1

$\Rightarrow \frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12} > 1$

Hence the result is true for n=1

Let us assume that the result holds for n=k.

That is

$$\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{3k+1} > 1$$

For n=k+1,

$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}$

$= \left[\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{3k+1}\right]+\frac{1}{3k+2}+\frac{1}{3k+3}\frac{1}{3k+4}-\frac{1}{k+1}$

$> 1+\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3}$

$\text{Now, if } 1+\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3} > 1$

Then we are through .or if

$\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3} > 0$

$\text{L.H.S} = \frac{(3k+4)(3k+3)+(3k+2)(3k+3)-2\,[(3k+2)(3k+4)]}{(3k+2)(3k+4)(3k+3)}$

$= \frac{3k+4-3k-2}{(3k+2)(3k+4)(3k+3)}$

which is positive. Hence the result is true for n=k+1.

Hence by mathematical induction, the result is true for all n.

Example: Using mathematical induction, show that

$1+\frac{1}{4}+\cdots+\frac{1}{n^2}>2-\frac{1}{n}$, for all natural numbers n greater than 1.

Solution: For n=2

$\text{L.H.S} = 1+\frac{1}{4}=\frac{5}{4}$ and $\text{R.H.S.} = 2-\frac{1}{2}=\frac{3}{2}$

$\frac{5}{4}<\frac{3}{2}$ Hence it holds for n=2.

Assume the result to hold for n=k

$\Rightarrow 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}<2-\frac{1}{k}$

For $n=k+1, \left[1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}\right]+\frac{1}{(k+1)^2}<2-\frac{1}{k}+\frac{1}{(k+1)^2}$

Now if we show that

$2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{(k+1)}$ or $\frac{1}{k}-\frac{1}{(k+1)^2}>\frac{1}{k+1}$ then we are through.

$\Rightarrow \frac{1}{k}-\frac{1}{(k+1)^2}-\frac{1}{k+1}>0 \Rightarrow \frac{(k+1)^2-k-k(k+1)}{k(k+1)^2}$

$= \frac{k^2+2k+1-k-k^2-k}{k(k+1)^2}=\frac{1}{k(k+1)^2}>0$

Hence the result is true for n=k+1.

Hence, by mathematical induction, the result is true for all n.

Example: Prove by mathematical induction $n!<\left(\frac{n+1}{2}\right)^n, n>1$

Solution: Let $P(n)=\left(\frac{n+1}{2}\right)^n>n!$

For m =2, L.H.S $=\left(\frac{3}{2}\right)^2=\frac{9}{4}$

R.H.S $=2!=2$

$\frac{9}{4}>2$, Hence P(2) is true.

Here $F(n)=\left(\frac{n+1}{2}\right)^n, G(n)=n!$

$F(n+1)=\left(\frac{n+2}{2}\right)^{n+1}$, $G(n+1)=(n+1)!$

Let P(n) is true i.e. F(n)>G(n)

$\Rightarrow \frac{F(n+1)}{F(n)}\frac{G(n)}{G(n+1)}=\frac{1}{2}\frac{(n+2)^{n+1}}{(n+1)^n}.\frac{n!}{(n+1)!}$

$=\frac{1}{2}\left(\frac{n+2}{n+1}\right)^{n+1}=\frac{1}{2}\left(1+\frac{1}{n+1}\right)^{n+1}>\frac{2}{2}=1 \left\{\left(1+\frac{1}{n+1}\right)^{n+1}\right\}$

$\Rightarrow \frac{F(n+1)}{G(n+1)}>\frac{F(n)}{G(n)}>1$

$\Rightarrow F(n+1)>G(n+1) \Rightarrow P(n+1)$ is true.

- **Problems based on Second Principle of induction:**

Example: It is given that $u_1 = 1, u_2 = 1$ and $U_{n+2} = U_{n+1} + U_n$ for $n \geq 1$.

Use mathematical induction to prove that $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Solution: For n=1, and 2, we have

$$u_1 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right] = 1$$

$$u_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = 1$$

$\Rightarrow$ The result is true for n =1,2

Assume the result to be true for $n \leq k$.

Then $u_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$

From the given relation, $u_{k+1} = u_k + u_{k-1}$

$$\Rightarrow U_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{1+\sqrt{5}}{2} + 1\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{1-\sqrt{5}}{2} + 1\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{3+\sqrt{5}}{2}\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{3-\sqrt{5}}{2}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\left(\frac{1+\sqrt{5}}{2}\right)^2\right] - \left(1 - \frac{\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Hence the result is true for n=k+1.

Hence, by mathematical induction, the result is true for all n.

Example: Prove by mathematical induction that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x} dx = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}, n \geq 1.$$

Solution: Let P(n)=$\int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x} dx = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$

For n=1, we find

$$\text{L.H.S.} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x} dx = \int_0^{\frac{\pi}{2}} \sin x\, dx = -[\cos x]_0^{\frac{\pi}{2}}$$

Thus P(1) is true.

Now assume P(m) be true , i.e.,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 mx}{\sin x} dx = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m-1}, n \geq 1. \qquad \text{.......(1)}$$

Further, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2(m+1)}{\sin x} dx - \int_0^{\frac{\pi}{2}} \frac{\sin^2 mx}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 2mx - \cos 2\,(m+1)x}{2 \sin x} dx$$

$$= -\frac{1}{2m+1} [\cos(2m+1)\,x]_0^{\frac{\pi}{2}} = \frac{-1}{2m+1} [0-1] = \frac{1}{2m+1}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin^2(m+1)}{\sin x} dx - \int_0^{\frac{\pi}{2}} \frac{\sin^2 mx}{\sin x} dx + \frac{1}{2m+1}$$

$$= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m-1} + \frac{1}{2m+1}$$

Hence is the required result.

Example: If $x + y = a + b$, $x^2 + y^2 = a^2 + b^2$, prove by Mathematical Induction that $x^n + y^n = a^n + b^n$ for all natural numbers n.

Solution: Let $P(n) \equiv x^n + y^n = a^n + b^n$

$P(1) \equiv x+y=a+b$ ....(1)

$P(2) \equiv x^2+y^2 = a^2+b^2$ ......(2)

Hence P(1) and P(2) are true. Assume the result to be true for $n \le k$.

$\Rightarrow x^{k-1} + y^{k-1} = a^{k-1} + b^{k-1}$ and $x^k + y^k = a^k + b^k$

In order to prove that P(k+1) is true, we write

$\Rightarrow x^{k+1} + y^{k+1} = x(a^k + b^k - y^k) + y(a^k + b^k - x^k)$

$= (a^k + b^k)(x + y) - xy(x^{k-1} + y^{k-1})$

$= (a^k + b^k)(a+b) - xy(x^{k-1} + {}^{yk-1})$

Now from (1) and (2) $xy = ab \Rightarrow x^{k+1} + y^{k+1} = a^{k+1} + b^{k+1}$

Which is the desired R.H.S for P(k+1).

Hence, by mathematical induction, the result is true for all n.

Example: For $x^3 = x + 1$, $a_n = a_{n-1} + b_{n-1}$, $b_n = a_{n-1} + b_{n-1} + c_{n-1}$, $c_n = a_{n-1} + c_{n-1}$, and $a_o = 0$, $b_o = 1$, $c_o = 0$ prove that $x^{3n} = a_n x + b_n + c_n x^{-1}$ $n \in N$.

Solution: We prove the result for n=1, first. According, we should have

$x^{3(1)} = a_1 x + b_1 + c_1 x^{-1}$. Also $a_1 = a_o + b_o = 0 + 1 = 1$

$b_1 = a_o + b_o + c_o = 0 + 1 + 0 = 1$

$c_1 = a_o + c_o = 0 + 0 = 0$

$\Rightarrow x^3 = 1.x + 1 = x + 1$, which is true.

Assume the result to be true for n= k $\Rightarrow x^{3k} = a_k x + b_k + c_k x^{-1}$

For n = k+1, $x^{3(k+1)} = x^{3k} . x^3$

$= (a_k x + b_k + c_k x^{-1})(x^3)$.

$= (a_k x + b_k + c_k x^{-1})(1+x)$ (since $x^3 = 1 + x$)

$= a_k x + a_k x^2 + b_k + b_k x + c_k x^{-1} + c_k$

$= x[a_k + b_k] + a_k x^{-1} x^3 + b_k + c_k x^{-1} + c_k$

$= x[a_k + b_k] + a_k x^{-1}(1+x) + b_k + c_k x^{-1} + c_k$ (since $x^3 = 1+x$)

$= x[a_k + b_k] + a_k x^{-1} + a_k + b_k + c_k x^{-1} + c_k$

Hence the result is true for n =k+1.

Hence, by mathematical induction, the result is true for all n.

3.2 BINOMIAL THEOREM

3.2.1 Introduction

Any algebraic expression consisting of only two terms is known as a **Binomial Expression**. Its expansion in power of x is known as the **Binomial Expansion**.

For example: (i) a + x (ii) $a^2 + 1/x^2$ (iii) 4x - 6y

Expression containing three terms are called trinomial. For example x+y+z is a trinomial expansion. In general an expression containing more than two terms is called multinomial.

3.2.1.1 Binomial Theorem

When n is a positive integer and a, x are two complex numbers, then

$(a + x)^n = {}^nC_0\, a^n x^0 + {}^nC_1\, a^{n-1}x^1 + {}^nC_2\, a^{n-2}x^2 + \cdots \ldots + {}^nC_r\, a^{n-r}x^r + \cdots \ldots + {}^nC_n\, x^n$ ……(i)

$= \sum_{r=0}^{n} {}^nC_r\, a^{n-r}x^r$

Where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, …..${}^nC_n$ are called Binomial coefficients, while (i) is called the Binomial Expansion.

Illustration: Expand $(x + 1/x)^7$.

Solution:

$$(x + 1/x)^7 = {}^7C_0x^7 + {}^7C_1x^6\ (1/x) + {}^7C_2x^5\,(1/x^2) + {}^7C_3x^4\,(1/x^3) + {}^7C_4x^3\,(1/x^4) + {}^7C_5x^2\ (1/x^5) + {}^7C_6x\,(1/x^6) + {}^7C_7\,(1/x^7)$$

$$= x^7 + 7x^5 + 21x^3 + (35\,x) + (35/x) + (21/x^3) + (7/x^5) + (1/x^7).$$

3.2.2 Properties of Binomial Expansion

- There are (n + 1) terms in the expansion of $(a + b)^n$, the first and the last term being a^n and b^n respectively. If ${}^nC_x = {}^nC_y$, then either x = y or x + y = n

  $\Longrightarrow {}^nC_r = {}^nC_{n-r} = \frac{n!}{r!(n-r)!}$.

- The general term in the expansion of $(a + x)^n$ is $(r + 1)^{th}$ term given as $T_{r+1} = {}^nC_r\, a^{n-r}x^r$

  Similarly the **general term** in the expansion of $(x + a)^n$ is given as $T_{r+1} = {}^nC_r\, x^{n-r}a^r$. The terms are considered from the beginning.

- The binomial coefficient in the expansion of $(a + x)^n$ which are equidistant from the beginning and the end are equal i.e. ${}^nC_r = {}^nC_{n-r}$.

Illustration: Find the expansion of $(a - x)^n$.

Solution: We know that

$$(a + x)^n = a^n + na^{n-1}(x) + \left(\frac{n(n-1)}{2!}\right)a^{n-2}(x)^2 + \cdots + (x)^n$$

Put x = -x in the above expansion, we get,

$$(a - x)^n = \big(a + (-x)\big)^n = a^n + na^{n-1}(-x) + \left(\frac{n(n-1)}{2!}\right)a^{n-2}(-x)^2 + \cdots + (-x)^n$$

$$= a^n - na^{n-1}x + \left(\frac{n(n-1)}{2!}\right)a^{n-2}\,x^2 - \cdots + (-1)^n x^n$$

Illustration: Find the value of $(a+\sqrt{(a^2-1)})^7 + (a-\sqrt{(a^2-1)})^7$.

Solution: Here, we have to find the sum of two expansions whose terms are numerically the same, but in the second expansion the second, fourth, sixth and eight terms are negative, and therefore cancel the corresponding terms of the first expansion. Hence, the given expression

$= 2\{a^7 + 21 a^5 (a^2 - 1) + 35 a^3(a^2 - 1)^2 + 7a (a^2 - 1)^3\}$

$= 2a (64 a^6 - 112 a^4 + 56 a^2 - 7)$

3.2.3 Binomial Coefficients

We know that,

$(a + x)^n = {}^nC_0\, a^n x^0 + {}^nC_1\, a^{n-1}x^1 + {}^nC_2\, a^{n-2}x^2 + \cdots + {}^nC_n\, x^n$ ... (i)

Let us write equation (i) in particular form by putting a = 1. Now it becomes,

$(1 + x)^n = {}^nC_0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \cdots + {}^nC_r\, x^r + \cdots + {}^nC_n\, x^n$ ... (ii)

Coefficients attached with different powers of x are called *Binomial Coefficients*.

Now, again put a = 1 in the expression,

$(a + x)^n = a^n + na^{n-1}(x) + \left(\frac{n(n-1)}{2!}\right) a^{n-2}(x)^2 + \cdots + \frac{(n(n-1)(n-2)\ldots.(n-r+1))}{r!}\, a^{n-r}\,(x)^r + \cdots + (x)^n$

we get,

$(1 + x)^n = 1 + nx + \left(\frac{n(n-1)}{2!}\right)x^2 + \cdots + \frac{(n(n-1)(n-2)\ldots.(n-r+1))}{r!}\, x^r + \cdots + x^n$ ...(iii)

If we compare coefficient of x^r in expansions (ii) and (iii), we have,

$${}^nC_r = \frac{(n(n-1)(n-2)\ldots.(n-r+1))}{r!} = \frac{n!}{r!\,(n-r)!}$$

Since expansion (ii) is valid for any value of x, we can replace x by 1/x , (x ≠ 0) in it, we get,

$\left(1 + \frac{1}{x}\right)^n = {}^nC_0 + {}^nC_1 \left(\frac{1}{x}\right)^1 + {}^nC_2 \left(\frac{1}{x}\right)^2 + \cdots + {}^nC_r \left(\frac{1}{x}\right)^r + \cdots + {}^nC_n \left(\frac{1}{x}\right)^n$

Multiplying on both sides by x^n, it becomes,

$(1 + x)^n = {}^nC_0 x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \cdots + {}^nC_r\, x^{n-r} + \cdots + {}^nC_n$ ... (iv)

Compare expansion (ii) and (iv). We observe that

${}^nC_r = {}^nC_{n-r}$, which implies that in the expansion of the equations similar to (ii) the r^{th} coefficient from the beginning is equal to the r^{th} coefficient from the end.

Note:

Put x = 1 in the expansion

$(1 + x)^n = {}^nC_0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \cdots + {}^nC_r\, x^r + \cdots + {}^nC_n\, x^n$, we get,

$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_r + \cdots + {}^nC_n$

When we take x = 1we obtain the desired result i.e. $\sum_{r=0}^{n} C_r = 2^n$.

Note: This one is very simple illustration of how we put some value of x and get the solution of the problem. It is very important how judiciously you exploit this property of binomial expansion.

Illustration: Find the value of $C_0 + C_2 + C_4 + ..$ in the expansion of $(1 + x)^n$.

Solution: We have,

$(1 + x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$

Now put x = -x;

$(1 - x)^n = {}^nC_0 - {}^nC_1 x^1 + {}^nC_2 x^2 - \cdots + (-1)^n \, {}^nC_n x^n$

Now, adding both expansions, we get,

$(1 + x)^n + (1 - x)^n = 2[{}^nC_0 + {}^nC_2 x^2 + {}^nC_4 x^4 + \cdots \ldots]$

Put x = 1

$\Rightarrow (1 + 1)^n + (1 - 1)^n = 2[{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots \ldots]$

$\Rightarrow \frac{2^n}{2} = C_0 + C_2 + C_4 + \ldots T_{r+1} = {}^nC_r a^{n-r} x^r$

or $C_0 + C_2 + C_4 + \cdots \ldots = 2^{n-1}$

3.2.4 Sum of Binomial Coefficients

Suppose we want to calculate the value of $\sum_{r=0}^{n} r\, C_r$ i.e., $0C_0 + 1\, C_1 + 2C_2 + \ldots + nC_n$

If we take a close look at the sum to be found, we find that coefficients are multiplied with respective powers of x.

If we want to multiply the coefficient of x by its power differentiation is of help. Hence differentiate both sides of

$(1 + x)^n = C_0 + C_1 x^1 + C_2 x^2 + \cdots + C_n x^n$, with respect to x we get

$n(1 + x)^{n-1} = 1\, C_1 x^{1-1} + 2C_2 x^{2-1} + \cdots + nC_n x^{n-1}$

Put x = 1, we get, n $2^{n-1} = 1\, C_1 + 2C_2 + \cdots + nC_n$

or $\sum_{r=0}^{n} r\, C_r = n2^{n-1}$, which is the answer.

Now, think how to find the following sum $\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_3}{3} + \cdots + \frac{C_n}{n+1}$

In this sum coefficients are divided by the respective power of x + 1. This expression can be achieved by Integrating the expansion of $(1 + x)^n$ under proper limits.

$$\int_0^1 (1+x)^n dx = \int_0^1 C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

$$\text{i.e.} \quad \frac{2^{n+1} - 1}{n+1} = \frac{C_0}{1} + \frac{C_1}{2} + \frac{C_3}{3} + \cdots + \frac{C_n}{n+1}$$

Note: It has been said earlier in this chapter that we should use and exploit the property that x can take any value in the expansion of $(1 + x)^n$.

Now, let us try to find the value of $C_0 - C_2 + C_4 - C_6 + \cdots$

Analyzing the above expression, we find that C_0, C_2, C_4 are all coefficients of even powers of x.

Had it been like $C_0 + C_2 + C_4 + \ldots\ldots$ we could have evaluated simply by the method described earlier or had it been like $C_0 - C_1 + C_2 - C_3 + \ldots$ we could have put x = -1 in the expansion of $(1 + x)^n$ and find the sum.

But here the case is different. The expression consists of coefficients of only even powers. In such cases when there is alternative sign charge for the coefficient, of same nature of powers (even or odd) use of i(iota) comes to our rescue.

$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots$

$(1 - x)^n = C_0 - C_1 x + C_2 x^2 + \ldots\ldots$

By adding $(1 + x)^n + (1 - x)^n = 2[C_0 + C_2 x^2 + C_4 x^4 + \ldots\ldots]$.

Clearly use of $\sqrt{-1}$ (iota) will generate the given expression.

$((1 + i)^n + (1 - i)^n)/2 = C_0 - C_2 + C_4 - C_6 + \cdots$

3.2.5 Coefficient of a Particular Term

3.2.5.1 General Term

In the expansion of $(a + x)^n$, the coefficient of second term is ${}^nC_1$, of the third term is ${}^nC_2$, of the fourth term is ${}^nC_3$ and so on. The suffix in each term being one less than the number of the term to which it applies; hence ${}^nC_r$ is the coefficient of the $(r+1)^{th}$ term. This is called the **General Term**, because by giving different numerical values to r any of the coefficients may be found. Also, the indices of a and x in the $(r + 1)^{th}$ term are expressible in terms of r.

Thus, General term of the expansion $(a + x)^n$ is denoted as

$$T_{r+1} = {}^nC_r\, a^{n-r}x^r = \left(\frac{n(n-1)(n-2)\ldots.(n-r+1)}{r!}\right) a^{n-r}x^r$$

General term is useful in finding the following in Binomial expansion.

(a) Particular term.
(b) Middle term
(c) Term independent of x.
(d) Term containing the greatest coefficient of x.

Illustration: Find the fifth term in the expansion of $(a + 2x^3)^{17}$.

Solution: General term of the expansion $(a + x)^n$ is

$T_{r+1} = {}^nC_r\, a^{n-r}x^r$

Now, for the given expression, required term is

$T_5 = {}^{17}C_4\, a^{13}(2x^3)^4$

$= 38080a^{13}x^{12}$.

Illustration: Find the term independent of x in $(1 + x + 2x^3)\,((3/2)x^2 - (1/3x))^9$.

Solution: We want to find the term independent of x in product of $(1 + x + 2x^3)$ and $((3/2)x^2 - (1/3x))^9$ .So, we should find the terms containing x^0, x^{-1} and x^{-3} in the expansion $((3/2)x^2 - (1/3x))^9$ and multiply them respectively with the appropriate term from $(1 + x + 2x^3)$

Now, $T_{r+1} = {}^nC_r\, a^{n-r}x^r$ [this is general term in $(a + x)^n$]

$= {}^9C_r((3/2)x^2)^{9-r}\,(-(1/3x))^r$

This is general term in $[((3/2)x^2 - (1/3x))^9]$

(i) to find the term containing x^0 we use $18 - 3r = 0$, i.e. $r = 6$

so coefficient of $x^0 = (-1)^6\ {}^9C_6\ 3^{-3}\ 2^{-3} = {}^9C_6\ (1/2^2.3^3)$.

(ii) to find the term containing x^{-1} we use $18-3r = -1$ i.e. $r = 19/3$ ($\notin$ I) which is impossible.

(iii) to find the term containing x^{-3} we put $18-3r = -3$ i.e. $r = 7$. So, coefficient of x^{-3} in $((3/2)x^2 - (1/3x))^9 = (-1)^7\ {}^9C_7\ 3^{-5}\ 2^{-2} = -{}^9C_7\ (1/2^2.3^5)$.

Therefore, required coefficient = ${}^9C_6\ (1/2^3.3^3)\times 1\text{-}2 \times {}^9C_7\ (1/2^2.3^5) = 17/54$.

Illustration: Find the coefficient of x^{24} in $(x^2 + 3a/x)^{15}$.

Solution: The general term ((r + 1)th term) is

$(x^2 + 3a/x)^{15} = {}^{15}C_r(x^2)^{15-r}\ (3a/x)^r = {}^{15}C_r\ x^{30-2r}\ (3^r a^r/x^r) = {}^{15}C_r\ 3^r\ a^r\ x^{30-3r}$

If this term contains x^{24}, then $30-3r = 24$

$\Rightarrow 3r = 6 \Rightarrow r = 2$.

Therefore, the coefficient of $x^{24} = {}^{15}C_2\ 9a^2$.

Illustration: If the binomial co-efficient of the $(2r + 4)^{th}$ term and the $(r - 2)^{th}$ term in the expansion of $(1 + x)^{18}$ are equal, find the value r.

Solution: The coefficient of $(2r + 4)^{th}$ term in $(1 + x)^{18} = {}^{18}C_{2r+3}$

and the coefficient of $(r-2)^{th}$ term = ${}^{18}C_{r-3}$, so that ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$.

Either $2r + 3 = r - 3, \Rightarrow r = -6$,

Or $2r + 3 + r - 3 = 18 \Rightarrow 3r = 18 \Rightarrow r = 6$. (Since $r \in N$)

Illustration: If the 4^{th} term in the expansion $(px + 1/x)^n$ is independent of x, find the value of n. Also calculate p if the 4^{th} term is 5/2.

Solution: Here $T_4 = T_{3+1} = {}^nC_3(px)^3\ (1/x)^{n-3} = {}^nC_3\ p^3\ x^{6-n}$.

T_4 is independent of $x \Rightarrow 6 - n = 0$ or $n = 6$.

Now given $T_4 = 5/2 \Rightarrow {}^6C_3.\ p^3 = 5/2 \Rightarrow p^3 = 5/2\ .\frac{1}{{}^6C_3}. = 1/8 \Rightarrow p = ½$

Illustration: If in the expansion of $(1 + x)^{43}$ the coefficient of $(2r + 1)^{th}$ term is equal to the coefficient of $(r + 2)^{th}$ term find r.

Solution: Given in the expansion of $(1 + x)^{43}$ the coefficient of $(2r + 1)^{th}$ term = the coefficient of $(r + 2)^{th}$ term.

${}^{43}C_{2r} = {}^{43}C_{r+1}$.

Either $2r = r + 1 \Rightarrow r = 1$,

or $2r + r + 1 = 43 \Rightarrow r = 14$.

Hence r =1, 14

Illustration: Find the coefficient of x^3 in the expansion of $(2x^2 - (1/3x))^9$.

Solution: T_{r+1} in $(2x^2 - (1/3x))^9$ is ${}^9C_r\ (2x^2)^{9-r}\ (-1/3x)^r = {}^9C_r 2^{9-r}\ (-1/3)^r\ x^{18-3r}$

$\Rightarrow$ Coefficient of x^3 is ${}^9C_5.2^4\ (-1/3)^5 = -224/27$.

3.2.5.2 Middle Term

(i) When n is even

Middle term of the expansion is the $\left(\frac{n}{2}+1\right)^{th}$ term

i.e., ${}^{n}C_{n/2}\, a^{n/2}\, x^{n/2}$ in the expansion of $(a + x)^n$.

(ii) When n is odd

Middle terms of the expansion are the $\left(\frac{n+1}{2}\right)^{th}$ term and the $\left(\frac{n+3}{2}\right)^{th}$term.

These are given by,

${}^{n}C_{\frac{(n-1)}{2}}\, a^{\frac{(n+1)}{2}}\, x^{\frac{(n-1)}{2}}$ and ${}^{n}C_{\frac{(n+1)}{2}}\, a^{\frac{(n-1)}{2}}\, x^{\frac{(n+1)}{2}}$ in the expansion of $(a + x)^n$.

Illustration: Find the middle term in the expansion of $(1 + x)^4$ and $(1 + x)^5$.

Solution: Expansion of $(1 + x)^4$have 5 terms, so third term is the middle term which is the $\left(\frac{4}{2}+ 1\right)^{th}$ term . Therefore, 3rd term is the middle term.

Expansion of $(1 + x)^5$ have 6 terms, so 3rd and 4th both are the middle terms, which are the $\left(\frac{5+1}{2}\right)^{th}$ and $\left(\frac{5+3}{2}\right)^{th}$ terms.

Note:

- The r^{th} term from the end = $(n - r + 2)^{th}$ term from the beginning.
- If there are two middle terms, then the binomial coefficients of two middle terms will be equal and those two coefficients will be greatest.

Illustration: Find the middle term in the expansion of $(1 - 2x + x^2)^n$.

Solution: We have $(1 - 2x + x^2)^n = [(1 - x)^2]^n = (1 - x)^{2n}$.

Here 2n is an even integer ⇒ $((2n/2) + 1)^{th}$ i.e. $(n+1)^{th}$ term will be the middle term.

Now $(n+1)^{th}$ term in $(1-x)^{2n}$ = ${}^{2n}C_n\ (1)^{2n-n}(-x)^n$ = ${}^{2n}C_n(-x)^n$ = $(2n)!/n!n!\ (-x)^n$.

3.2.6 Greatest Binomial Coefficient

To determine the greatest coefficient in the binomial expansion, $(1+x)^n$, when n is a positive integer. Coefficient of

$$\frac{T_{r+1}}{T_r} = \frac{C_r}{C_{r-1}} = \frac{n-r+1}{r} = \left(\frac{n+1}{r}\right) - 1.$$

Now the $(r+1)^{th}$ binomial coefficient will be greater than the r^{th} binomial coefficient when, $T_{r+1} > T_r$

$$\Rightarrow \left(\left(\frac{n+1}{r}\right) - 1\right) \geq 1$$

$$\Rightarrow \left(\frac{n+1}{2}\right) \geq r \qquad \text{....... (1)}$$

But r must be an integer, and therefore when n is even, the greatest binomial coefficient is given by the greatest value of r, consistent with (1) i.e., r = n/2 and hence the greatest binomial coefficient is ${}^{n}C_{n/2}$.

Similarly if n be odd, the greatest binomial coefficient is given when,

r = (n-1)/2 or (n+1)/2 and the coefficient itself will be ${}^{n}C_{n+1/_2}$ or ${}^{n}C_{n-1/_2}$ both being equal.

Note: The greatest binomial coefficient is the binomial coefficient of the middle term.

Illustration: Show that the greatest coefficient in the expansion of $(x + 1/x)^{2n}$ is

(1.3.5...(2n-1).2n)/n!.

Solution: Since middle term has the greatest coefficient.

So, greatest coefficient = coefficient of middle term

$$= {}^{2n}C_n = (1.2.3...2n)/n!\,n! = (1.3.5...(2n-1).2n)/n!.$$

Numerically greatest term

To determine the numerically greatest term in the expansion of $(a + x)^n$, where n is a positive integer.

Consider $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^{n}_{r}C\, a^{n-r}x^{r}}{{}^{n}_{r-1}C\, a^{n-r+1}x^{r-1}}\right| = \left|\frac{{}^{n}_{r}C}{{}^{n}_{r-1}C}\right|\left|\frac{x}{a}\right| = \left|\frac{n-r+1}{r}\right|.\left|\frac{x}{a}\right| = \left|\frac{n+1}{r} - 1\right|\left|\frac{x}{a}\right|.$

Thus

$$|T_{r+1}| > |T_r| \ \text{ if } \left\{\frac{n+1}{r} - 1\right\}\left|\frac{x}{a}\right| > 1$$

$$\text{i.e.}, \frac{n+1}{r} > 1 + \left|\frac{a}{x}\right| \Rightarrow \frac{n+1}{1+\left|\frac{a}{x}\right|} > r. \qquad ...(1)$$

Note:

$\left\{\left(\frac{n+1}{r}\right) - 1\right\}$ must be positive since n > r.

Thus T_{r+1} will be the greatest term if, r has the greatest value as per the equation (1).

Illustration: Find the greatest term in the expansion of $(3-2x)^9$ when x = 1.

Solution: $\frac{T_{r+1}}{T_r} = \left(\frac{9-r+1}{r}\right)\left(\frac{2x}{3}\right) > 1$

i.e. 20 > 5r

If r = 4, then $T_{r+1} = T_r$ and these are the greatest terms. Thus 4^{th} and 5^{th} terms are numerically equal and greater than any other term and their value is equal $3^5 \times {}^{9}C_4 \times (-2)^4 = 489888$

Illustration: Find the greatest term in the expansion of $(2 + 3x)^9$ if x = 3/2.

Solution: Here $\frac{T_{r+1}}{T_r} = \left(\frac{n-r+1}{r}\right)\left(\frac{3x}{2}\right)$

= ((10-r)/r)(3x/2), (where x = 3/2)

= ((10-r)/r)(3/2)(3/2) = ((10-r)/r).9/4 = (90-9r)/4r

Therefore $T_{r+1} \geq T_r$, if 90 - 9r ≥ 4r

⇒ 90 ≥ 13r ⇒ r ≤ 90/13 and r being an integer, r = 6.

Hence $T_{r+1} = T_7 = T_{6+1} = {}^9C_6\,(2)^3\,(3x)^6 = 3^{13}.7/2$.

Illustration: Given that the 4^{th} term in the expansion of $(2 + (3/8)x)^{10}$ has the maximum numerical value, find the range of values of x for which this will be true.

Solution: Given 4^{th} term in $(2 + (3/8)x)^{10} = 2^{10}\,(1 + (3/16)x)^{10}$, is numerically greatest

$\Rightarrow \left|\frac{T_4}{T_3}\right| \geq 1 \text{ and } \left|\frac{T_5}{T_4}\right| \leq 1$

$\left|\frac{{}^{10}_{3}C}{{}^{10}_{2}C}.\frac{3}{16}X\right| \geq 1 \text{ and } \left|\frac{{}^{10}_{3}C}{{}^{10}_{2}C}.\frac{3}{16}X\right| \leq 1$

$\Rightarrow\ |x|\ \geq\ 2 \text{ and } |x|\ \leq\ 64/21$

$\Rightarrow x\ \varepsilon\ [-(64/21),-2]\cup[2,(64/21)]$.

Particular Cases

We have $(a\ +\ x)^n = a^n + {}^nC_1a^{n-1}(x) + {}^nC_2a^{n-2}(x)^2 + \cdots + (x)^n$ ...... (1)

(i) Putting x = -x in (1), we get

$(a-\ x)^n = a^n - {}^nC_1a^{n-1}(x) + {}^nC_2a^{n-2}(x)^2 + \cdots + (-1)^r\,{}^nC_ra^{n-r}(x)^r + \cdots + (-1)^n(x)^n$

(ii) Putting a = 1 in (1), we get,

$(1\ +\ x)^n = {}^nC_0\ + {}^nC_1\,x^1 + {}^nC_2\,x^2 + \cdots + {}^nC_rx^r + \cdots + {}^nC_n\,x^n$ ... (A)

(iii) Putting a = 1, x = -x in (1), we get

$(1-\ x)^n = {}^nC_0 - {}^nC_1\,x^1 + {}^nC_2\,x^2 - \cdots + (-1)^r\,{}^nC_rx^r + \cdots \ldots.(-1)^n\,{}^nC_n\,x^n$ ... (B)

Points to Remember:

- $\frac{T_{r+1}}{T_r} = \left(\frac{n-r+1}{r}\right).\left(\frac{x}{a}\right)$ for the binomial expansion of $(a + x)^n$.
- ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$.
- $r\ {}^nC_r = n\ {}^{n-1}C_{r-1}$
- $\frac{{}^{n}_{r}C}{r+1} = \frac{{}^{n+1}_{r+1}C}{n+1}$
- When n is even,

  $(x+a)^n + (x-a)^n = 2(x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \cdots + {}^nC_na^n)$.
- When n is odd,

  $(x+a)^n + (x-a)^n = 2(x^n + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_{n-1}xa^{n-1})$.
- When n is even

  $(x+a)^n - (x-a)^n = 2({}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots + {}^nC_{n-1}x\,a^{n-1})$.
- When n is odd

  $(x+a)^n - (x-a)^n = 2({}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots + {}^nC_n\,a^n)$.

3.2.7 Properties of Binomial Coefficients

For the sake of convenience, the coefficients ${}^nC_o$, ${}^nC_1$, ..., ${}^nC_r$, ..., ${}^nC_n$ are usually denoted by C_o, C_1,..., C_r, ..., C_n respectively

Put x = 1 in (A) and we get, $2^n = C_o + C_1 + ... + C_n$ ... (D)

Also putting x = -1 in (A) we get,

$0 = C_o - C_1 + C_2 - C_3 + ...$

$\Rightarrow C_0 + C_2 + C_4 + ... + C_1 + C_3 + C_5 + ... = 2^n$.

Hence $C_o + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$.

Illustration: If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then prove that

$C_0 + (C_o + C_1) + (C_0 + C_1 + C_2) + ... (C_0 + C_1 + C_2 + ... + C_{n-1}) = n2^{n-1}$

(where n is an even integer.

Solution: $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + ... (C_0 + C_1 + C_2 + ... + C_{n-1})$

$=C_0+(C_0 + C_1 + C_2 + ... + C_{n-1})+(C_0 + C_1)+(C_0 + C_1 + C_2 + ... + C_{n-2})+...$

$= (C_0 + C_1 + C_2 + ... + C_n)+ (C_0 + C_1 + C_2 + ... + C_n)+...$ n/2 times

$= (n/2)2^n = n.\ 2^{n-1}$

3.2.8 Some Important Results

1. Differentiating $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ of both sides we have,

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_nx^{n-1}$. ... (E)

Put x = 1 in (E) so that $n2^{n-1} = C_1 + 2C_2 + 3C_3 + ... + nC_n$.

Put x = -1 in (E) so that $0 = C_1 - 2C_2 + ... + (-1)^{n-1} nC_n$.

Differentiating (E) again and again we will have different results.

2. Integrating $(1 + x)^n$, we have,

$((1+x)^{n+1})/(n+1) + C = C_0x + C_1x^2/2 + C_2x^3/3 + ... + C_nx^{n+1}/(n+1)$

(where C is a constant)

For x = 0, we get C = -1/(n+1).

Therefore $((1+x)^{n+1} - 1)/(n+1) = C_0x + C_1x^2/2 + C_2x^3/3 + ... + C_nx^{n+1}/(n+1)$ ..... (F)

Put x = 1 in (F) and get

$(2^{n+1} - 1)/n+1= C_0 + C_1/2 + ...C_n/n+1$.

Put x = -1 in (F) and get, $1/(n+1) = C_0 - C_1/2 + C_2/3 -$ ......

Put x = 2 in (F) and get, $(3^{n+1}-1)/n+1 = 2\ C_0 + 2^2/2\ C_1 + 2^3/3\ C_2 + ... + 2^{n+1}/n+1\ C_n$.

- Problems Related to Series of Binomial Coefficients in which each term is a Product of an Integer and a Binomial Coefficient, i.e. in the form K. ${}^nC_r$.

Illustration: If $(1 + x)^n = \sum_{r=0}^{n} C_r\, x^r$ Then prove that $C_1 + 2C_2 + 3C_3 + ... + nC_n = n2^{n-1}$.

Solution: **Method (i): r^{th} term of the given series**

r^{th} term of the given series, $t_r = {}^nC_r$

$\Rightarrow t_r = r \times \frac{n}{r} \times {}^{n-1}C_{r-1} = n \times {}^{n-1}C_{r-1}$

(Because ${}^nC_r = \frac{n}{r} \times {}^{n-1}C_{r-1}$)

Sum of the series = $\sum_{r=1}^{n} t_r = n \sum_{r=1}^{n} {}^{n-1}_{r-1}C$

Put x = 1 in the expansion of $(1 + x)^{n-1}$, so that

$({}^{n-1}C_0 + {}^{n-1}C_1 + ... + {}^{n-1}C_{n-1}) = 2^{n-1}$

$\Rightarrow \sum_{r=1}^{n} t_r = n \sum_{r=1}^{n} {}^{n-1}_{r-1}C = n2^{n-1}$

Method (ii): By Calculus

We have $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$. ... (1)

Differentiating (1) w.r.t. x, we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3 x^2 + ... + n\, C_nx^{n-1}$. ... (2)

Putting x = 1 in (2), we have, $n\, 2^{n-1} = C_1 + 2C_2 + ... + nC_n$. ... (3)

Illustration: If $(1 + x)^n = \sum_{r=0}^{n} C_r\, x^r$ then prove that $C_0+2.C_1+3.C_2+...+(n+1)C_n=2^{n-1}(n+2)$.

Solution: **Method (i): r^{th} term of the given series**

rth term of the given series

$t_r = {}^nC_{r-1} = [(r-1) + 1].\, {}^nC_{r-1}$

$= (r-1)\, {}^nC_{r-1} + {}^nC_{r-1} = n.\, {}^{n-1}C_{r-2} + {}^nC_{r-1}$ (because ${}^nC_{r-1} = n/(r-1).\, {}^{n-1}C_{r-2}$)

Sum of the series = $\sum_{r=1}^{n+1} t_r = n \sum_{r=1}^{n+1} ({}^{n-2}_{r-2}C) + \sum_{r=1}^{n+1} ({}^{n}_{r-1}C)$

$= n[{}^{n-1}C_0 + {}^{n-1}C_1 + ... + {}^{n-1}C_{n-1}] + [{}^nC_0 + {}^nC_1 + ... + {}^nC_n] = n.2^{n-1} + 2n$
$= 2^{n-1}\, (n+2)$.

Method (ii) by Calculus

We have $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$. ... (1)

Multiplying (1) with x, we get

$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_nx^{n+1}$. ... (2)

Differentiating (2) w.r.t. x, we have

$(1 + x)^n + n(1 + x)^{n-1}\, x = C_0 + 2C_1x + ... + (n+1)C_nx^n$ ... (3)

Putting x = 1 in (3), we get

$2^n + n.2^{n-1} = C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n$

$\Rightarrow C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = 2^{n-1}\, (n+2)$.

- Problems Related to Series of Binomial Coefficient in Which Each Term is Binomial Coefficient divided by an Integer, i.e. in the Form of ${}^nC_r/k$.

Illustration: If $(1 + x)^n = \sum_{r=0}^{n} C_r\, x^r$,

show that $C_0 + \frac{C_2}{2} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Solution: **Method (i): r^{th} term of the given series**

$t_r = \frac{{}^{n}_{r-1}C}{r} = \frac{n+1}{r}$,

${}^nC_{r-1} \frac{1}{n+1} = \frac{1}{n+1}\ {}^{n+1}C_r$

Sum of the series $= \sum_{r=1}^{n+1} t_r$

$= n \sum_{r=1}^{n+1} \frac{{}^{n+1}_{r}C}{n+1} = \frac{1}{n+1} ({}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1})$

$= \frac{1}{n+1}\left({}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} + - {}^{n+1}C_0\right) = \frac{2^{n+1}-1}{n+1}$

Method (ii): By Calculus

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ ... (1)

Integrating both the sides of (1) w.r.t. x between the limits 0 to x, we get

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \cdots + \frac{C_nx^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^2}{3} + \cdots + \frac{C_nx^{n+1}}{n+1} \quad \ldots (2)$$

Substituting x = 1 in (2), we get $\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_2}{2} + \cdots + \frac{C_n}{n+1}$

Illustration: If $(1+x)^n = \sum_{r=0}^{n} C_r\, x^r$ Show that $\frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \cdots + \frac{C_n}{n+2} = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$

Solution: **Method I: r^{th} term of the given series,** T_r **=** $\frac{{}^nC_{r-1}}{r+1}$

$= \frac{1}{n+1}\left({}^{n+1}_{\ \ r}C - \frac{1}{n+2}\,{}^{n+2}_{r+1}C\right)$

Sum of the series $\sum_{r=1}^{n+1} \frac{1}{n+1}\left({}^{n+1}_{\ \ r}C - \frac{1}{n+2}\,{}^{n+2}_{r+1}C\right)$

$= \frac{1}{n+1}(2^{n+1} - 1) - \frac{1}{(n+1)(n+2)}(2^{n+2} - {}^{n+2}_{\ \ 1}C - {}^{n+2}_{\ \ 0}C) = \frac{n,2^{n+1}+1}{(n+1)(n+2)}$

Method II: (By Calculus)

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

$\Rightarrow x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + \ldots + C_nx^{n+1}$ ... (1)

Integrating both the sides of (1) with respect to x

$$\Rightarrow \frac{x(1+x)^{n+1}}{n+1} - \int\left(\frac{(1+x)^{n+1}}{n+1}\right)dx = \frac{x^2}{2}C_0 + \frac{x^3}{3}C_1 + \frac{x^4}{4}C_2 + \cdots + \frac{x^{n+2}}{n+2}C_n$$

$$\Rightarrow \frac{x(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+1)(n+2)} + k = \frac{x^2}{2}C_0 + \frac{x^3}{3}C_1 + \frac{x^4}{4}C_2 + \cdots + \frac{x^{n+2}}{n+2}C_n$$

Put x = 0, $\Rightarrow$ k = 1/((n+1)(n+2))

$$\Rightarrow \frac{x(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{x^2}{2}C_0 + \frac{x^3}{3}C_1 + \frac{x^4}{4}C_2 + \cdots + \frac{x^{n+2}}{n+2}C_n$$

Put x = 1,

$$\Rightarrow \frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \cdots + \frac{C_n}{n+2} = \frac{n.2^{n+1}+1}{(n+1)(n+2)}.$$

- Problem Related to Series of Binomial Coefficients in Which Each Term is a Product of two Binomial Coefficients.

(a) If sum of lower suffices of binomial expansion in each term is the same

i.e. ${}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + \ldots + {}^nC_n\,{}^nC_0$

i.e. 0 + n = 1 + (n-1) = 2 + (n-2) = n + 0.

Then the series represents the coefficients of x^n in the multiplication of the following two series

$(1+x)^n = C_0 + C_1x + C_2x^2 +...+ C_nx^n$

and $(1+x)^n = C_0 + C_1x + C_2x^2 +...+ C_nx^n$.

Illustration: Prove that $C_0C_r + C_1C_{r+1} + C_2C_{r+2} +...+ C_{n-r}\, C_n = (2n)!/((n-r)!(n+r)!)$.

Solution: We have,

$C_0 + C_1x + C_2x^2 + ... + C_nx^n = (1+x)^n$ ... (1)

Also $C_0x^n + C_2x^{n-2} +...+ C_n = (x+1)^n$ ... (2)

Multiplying (1) and (2), we get

$(C_0 + C_1x +...+ C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} +...+ C_n) = (1+x)^{2n}$ ... (3)

Equating coefficient of x^{n-r} from both sides of (3), we get

$C_0C_r + C_1C_{r+1} + C_2C_{r+2} +...+ C_{n-r}C_n = 2nC_{n-r} = (2n)!/((n-r)!(n+r)!)$.

Illustration: Prove that $C_o^2 + C_1^2 +...+ C_n^2 = (2n)!/n!n!$.

Solution: Since

$(1 + x)^n = C_0 + C_1x + C_2x^2 +...+ C_nx^n,$ ... (1)

$(x + 1)^n = C_0x^n + C_1x^{n-1} +...+ C_n,$ ... (2)

$(C_0 + C_1x + C_2x^2 +...+ C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} +...+ C_n)=(1+x)^{2n}$.

Equating coefficient of x^n, we get

$C_0^2 + C_1^2 + C_2^2 +...+ C_n^2 = {}^{2n}C_n = (2n)!/n!n!$.

Illustration: If $(1 + x)^n = \sum_{r=0}^{n} {}^nC_rx^r$

then prove that ${}^mC_r\,{}^nC_0 + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 +...+ {}^mC_1\,{}^nC_{r-1} + {}^mC_0\,{}^nC_r = {}^{m+n}C_r$

where m, n, r are positive integers and $r < m$ and $r < n$.

Solution:

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 +...+ {}^nC_rx^r +...+ {}^nC_nx^n$ ... (1)

and also

$(1+x)^m = {}^mC_0 + {}^mC_1x + {}^mC_2x^2 +...+ {}^mC_rx^r +...+ {}^mC_mx^m$ ... (2)

Multiplying (1) and (2), we get

$({}^nC_0 + {}^nC_1x + {}^nC_2x^2 +...+ {}^nC_rx^r +...+ {}^nC_nx^n)x$

$({}^mC_0 + {}^mC_1x + {}^mC_2x^2 +...+ {}^mC_rx^r +...+ {}^mC_mx^m) = (1+x)^{m+n}$

$= {}^{m+n}C_0 + {}^{m+n}C_1x + {}^{m+n}C_2x^2 +...+ {}^{m+n}C_rx^r +...+ {}^{m+n}C_{m+n}x^{m+n}$

Equating the coefficient of x^r, we get

${}^mC_r\,{}^nC_0 + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 +...+ {}^mC_1\,{}^nC_{r-1} + {}^mC_0\,{}^nC_r = {}^{m+n}C_r$

(b) If one series has constant lower suffices and other has varying lower suffice

Illustration: Prove that ${}^nC_0.{}^{2n}C_n$ - ${}^nC_1{}^{2n-2}C_n$ + ${}^nC_2{}^{2n-4}C_n$ -...= 2^n.

Solution: ${}^nC_0.{}^{2n}C_n - {}^nC_1{}^{2n-2}C_n + {}^nC_2{}^{2n-4}C_n$ -...

= coefficient of x^n in$[{}^nC_0(1+x)^{2n} - {}^nC_1(1+x)^{2n-2}+{}^nC_2(1+x)^{2n-4} - ...]$

= coefficient of x^n in

$[{}^nC_0((1+x)^2)^n - {}^nC_1((1+x)^2)^{n-1} + {}^nC_2((1+x)^2)^{n-2} - ...]$

= coefficient of x^n in $[(1+x)^2 - 1]^n$

= coefficient of x^n in $(2x+x^2)^n$ = co-efficient of x^n in $x^n\ (2+x)^n = 2^n$.

3.2.9 Application of Binomial Expansion

- **Divisibility problems:**

Let $(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 +...+ {}^nC_nx^n$.

In any divisibility problem, we have to identify x and n. The number by which division is to be made can be, x, x^2 or x^3, but the number in the base is always expressed in form of (1+x).

Illustration: Find the remainder when 7^{103} is divided by 24.

Solution: $7^{103} = 7(50 - 1)^{51} = 7(50^{51} - {}^{51}C_1\ 50^{50} + {}^{51}C_2\ 50^{49} - ... - 1)$

$= 7(50^{51} - {}^{51}C_1 50^{50} +...+ {}^{51}C_{50}50) - 7 - 18 + 18$

$= 7(50^{51} - {}^{51}C_1 50^{50} +...+ {}^{51}C_{50}50) - 25 + 18$

$\Rightarrow$ Remainder is 18.

- **Problems involving the greatest integer function.**

The question generally involves working with binomial expression on surds.

Illustration: If I is the integral part and f is the fraction part of $(2 + \sqrt{3})^n$ then prove that (I + f)(1- f) = 1. Also prove that I is an odd integer.

Solution: $(2 + \sqrt{3})^n = I + f$ where I is an integer and $0 < f < 1$.

We have to show that I is odd and that (I + f)(1 - f) = 1

Here $(2 + \sqrt{3})^n\ (2 - \sqrt{3})^n = (4 - 3)^n = 1$

$\therefore (2 + \sqrt{3})^n(2 - \sqrt{3})^n=1$. It is thus required to prove that $(2 + \sqrt{3})^n=1-f$

But, $(2 + \sqrt{3})^n + (2 - \sqrt{3})^n$

$= [2^n + C_1.2^{n-1}.\sqrt{3} + C_2 2^{n-2}.(\sqrt{3})^2 + ...] + [2^n - C_1.2^{n-1}.\sqrt{3} + C_2 2^{n-2}.(\sqrt{3})^2 - ...]$

$=2[2^n + C_2.2^{n-2}.3 + C_4 2^{n-4}.3^2 +...]$ = even integer

$\therefore$ Now $0 < (2 - \sqrt{3}) < 1$

$\therefore\ 0 < (2 - \sqrt{3})^n < 1$

$\therefore$ If $(2 - \sqrt{3})^n = f'$ then I + f + f' = Even

Now $0 < f < 1$ and $0 < f' < 1$ ... (1)

$\therefore$ f + f' = integer

(1) and (2) imply that $f + f' = 1 (\because 0 < f + f' < 2)$

$\therefore$ I is odd and $f' = 1 - f$

$\Rightarrow (I + f)(1 - f) = 1$.

- **Binomial expansion with non- positive exponent**

The characteristics discussed above were confined to positive integer n. If n takes any other value, then binomial theorem is written as:

Say, we have to find out $(x+a)^n$ $n \notin I^+$

If a > 1 then it is written as $a^n (1+ (x/a))^n$. This is expanded as

$a^n (1+(x/a))^n = a^n[1+n(x/a) + (n(n-1)/2!) (x/a)^2 + (n(n-1)(n-2))/3! (x/a)^3 + ...\infty]$

Since n is not positive integer therefore the series on the right hand side will converge only for $|x/a|<1$. Moreover, there are infinite terms in the expansion contrary to the binomial expansion for a positive integer n.

- **Multinomial Expansion**

If such a case arises, then it is not called Binomial Expansion, it is called Multinomial Expansion. If $n \varepsilon N$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + ... + x_k)^n$ is $\left(\frac{n!}{a_1!a_2!....a_k!}\right)\left(x_1^{a_1}.x_2^{a_2}.......x_k^{a_k}\right)$,where $a_1 + a_2 + a_3 + ... + a_k = n$ and $a < a_i < n$, I = 1, 2, 3, ... k and the total number of terms in the expansion is ${}^{n+k-1}C_{n-1}$.

3.2.10 Solved Examples

Example: Find the coefficient of the term independent of x in expansion of $(3x -(2/x^2))^{15}$.

Solution: The general term of $(3x - (2/x^2))^{15}$ is written, as $T_{r+1} = {}^{15}C_r (3x)^{15-r} (-2/x^2)^r$.

It is independent of x if,

$15 - r - 2r = 0 \Rightarrow r = 5$

$\therefore \quad T_6 = {}^{15}C_5(3)^{10}(-2)^5 = - {}^{15}C_5 3^{10} 2^5$.

Example: If the coefficient of $(2r + 4)^{th}$ and $(r - 2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal then find the value of r.

Solution: The general term of $(1 + x)^n$ is $T_{r+1} = C_r x^r$

Hence coefficient of $(2r + 4)^{th}$ term will be

$T_{2r+4} = T_{2r+3+1} = {}^{18}C_{2r+3}$

and coefficient or $(r - 2)^{th}$ term will be

$T_{r-2} = T_{r-3+1} = {}^{18}C_{r-3}$. $\Rightarrow {}^{18}C_{2r+3} = {}^{18}C_{r-3}$.

$\Rightarrow (2r + 3) + (r-3) = 18$ $(\because {}^nC_r = {}^nC_K \Rightarrow r = k$ or $r + k = n)$

$\therefore \quad r = 6$

Example: If a_1, a_2, a_3 and a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$ then prove that: $a_1/(a_1+a_2) + a_3/(a_3+a_4) = 2a_2/(a_2+a_3)$

Solution: As a_1, a_2, a_3 and a_4 are coefficients of consecutive terms, then Let $a_1 = {}^nC_r$

$a_2 = {}^nC_{r+1}$

$a_3 = {}^nC_{r+2}$ and $a_4 = {}^nC_{r+3}$

Now $a_1/(a_1+a_2) = {}^nC_r/({}^nC_{r+}{}^nC_{r+1}) = 1/(1+((n-r)/(r+1))) = (r+1)/(n+1)$

Similarly, $a_3/(a_3+a_4) = (r+3)/(n+1)$

Now $a_3/(a_3+a_4) + a_1/(a_1+a_2) = (2r+4)/(n+1)$

$= 2(r+2)/(n+1) = 2a_2/(a_2+a_3)$

Example: Find out which one is larger $99^{50} + 100^{50}$ or 101^{50}.

Solution: Let's try to find out $101^{50} - 99^{50}$ in terms of remaining term i.e.

$101^{50} - 99^{50} = (100+1)^{50} - (100 - 1)^{50}$

$= (C_0.100^{50} + C_1 100^{49} + C_2.100^{48} + ...) - (C_0 100^{50} - C_1 100^{49} + C_2 100^{48} - ......)$

$= 2[C_1.100^{49} + C_3 100^{47} + ...]$

$= 2[50.100^{49} + C_3 100^{47} + ...]$

$= 100^{50} + 2[C_3 100^{47} + ...] > 100^{50}$

$\Rightarrow 101^{50} > 99^{50} + 100^{50}$

Example: Find the value of the greatest term in the expansion of $\sqrt{3}(1+(1/\sqrt{3}))^{20}$.

Solution: Let T_{r+1} be the greatest term, then $T_r < T_{r+1} > T_{r+2}$

Consider: $T_{r+1} > T_r$

$\Rightarrow {}^{20}C_r \ (1/\sqrt{3})^r > 20C_{r-1}(1/\sqrt{3})^{r-1}$

$\Rightarrow ((20)!/(20-r)!r!)\ (1/(\sqrt{3})^r) > ((20)!/(21-r)!(r-1)!)\ (1/(\sqrt{3})^{r-1})$

$\Rightarrow r < 21/(\sqrt{3}+1)$

$\Rightarrow r < 7.686$ .... (i)

Similarly, considering $T_{r+1} > T_{r+2}$

$\Rightarrow r > 6.69$ .... (ii)

From (i) and (ii), we get r = 7

Hence greatest term = T_8 = 25840/9

Example: Find the coefficient of x^{50} in the expansion of $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2 (1+x)^{998} + ... + 1001x^{1000}$.

Solution: Let $S = (1 + x)^{1000} + 2x(1+x)^{999} + ... + 1000x^{999} (1+x) + 1001 x^{1000}$

This is an Arithmetic Geometric Series with r = x/(1+x) and d = 1.

Now

$(x/(1+x))S = x(1 + x)^{999} + 2x^2 (1 + x)^{998} + ... + 1000x^{1000} + 1001x^{1001}/(1+x)$

Subtracting we get,

$(1 - (x/(x+1)))\ S = (1+x)^{999} + 2x(1+x)^{998} + ... + x^{1000} - 1001x^{1000}/(1+x)$

or $S = (1+x)^{1001} + x(1+x)^{1000} + x^2(1+x)^{999} + ... + x^{1000} (1+x) - 1001x^{1001}$

This is G.P. and sum is

$S = (1+x)^{1002} - x^{1002} - 1002x^{1001}$

So the coefficient of x^{50} is = ${}^{1002}C_{50}$

Example: Show that $\sum_{k=0}^{n} {}^{n}C_k(\sin kx)\cos(n-k)x = 2^{n-1}\sin(nx)$

Solution: We have $\sum_{k=0}^{n} {}^{n}C_k(\sin kx)\cos(n-k)x$

$=\frac{1}{2}\sum_{k=0}^{n} {}^{n}C_k[(\sin(kx+nx-kx)+\sin(kx-nx+kx))]$

$=\frac{1}{2}\sum_{k=0}^{n} {}^{n}C_k \sin nx + \frac{1}{2}{}^{n}C_k(\sin\sum_{k=0}^{n} 2kx - nx)$

$$= \frac{1}{2}\sin n\,x\sum_{k=0}^{n} {}^{n}C_k\ \frac{1}{2}\begin{bmatrix} {}^{n}C_0(\sin(-nx)) + {}^{n}C_0(\sin\big((2-n)x\big) + \cdots \\ +{}^{n}C_{n-1}(\sin\big((n-2)x\big) + {}^{n}C_n(\sin(nx)) \end{bmatrix}$$

$= 2^{n-1}\sin(nx)$

(as terms in bracket, which are equidistant, from end and beginning will cancel each other).