Vector Algebra

Scalars.

A quantity which has only magnitude but is not associated to definite direction is called a scalar quantity.

Vectors.

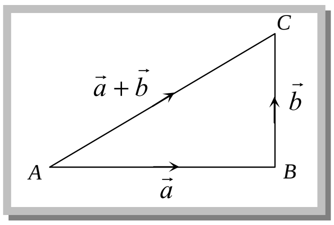
A quantity which has both magnitude as well as direction is called a vector quantity.

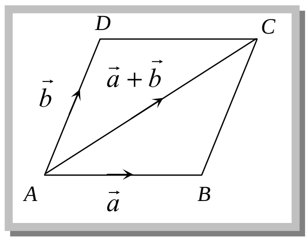
Addition of two vectors $\vec{a}$ and $\vec{b}$.

(i) **Sum of two vectors by law of triangle** : If two vectors are represented by the two sides of a triangle taken in order, then their sum is represented by the third side of the triangle taken in opposite direction.

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$$\bar{a}+\bar{b}=\bar{a}+\bar{b}$$

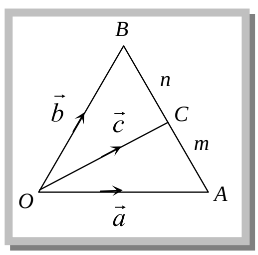




(ii) **Sum of two vectors by law of parallelogram** : It states that if ABCD is a parallelogram, then the sum of vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is the vectors $\overrightarrow{AC}$.

$$\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

Section formula.



(1) **Internal division** : If $\vec{a}$ and $\vec{b}$ are the position vector of two points A and B, then the point C which divides AB in the ratio $m : n$, where m and n are positive real number has the position vector $\vec{c}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$

(2) **External division :** $\vec{c} = \frac{m\vec{b} - n\vec{a}}{m - n}$

Note : ❑ If $m = n$, then *C* is the mid point of AB and $\vec{c} = \frac{\vec{a} + \vec{b}}{2}$

Centroid of triangle.

If $A(\vec{a}),\ B(\vec{b}),\ C(\vec{c})$ are the vertices of a triangle *ABC*, then centroid of triangle *ABC* is, $\vec{G} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$.

Linear combination of vecotrs.

(1) A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a},\ \vec{b},\ \vec{c}$........., if these exist and scalars x, y, z.......... such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \text{..........}$.

(2) **Linear dependent vectors** : $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 \text{........} + x_n\vec{a}_n = \vec{0}$, Where, $x_1, x_2, x_3 \text{..........}, x_n \neq 0$

(3) **Linear independent vectors:** $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 \text{........} + x_n\vec{a}_n = \vec{0}$,

Where, $x_1, x_2, x_3 \text{..........}, x_n = 0$ not all zero.

Some important results.

(1) If $\vec{a},\ \vec{b}$ be two non- zero, non collinear, vectors and *x*, *y* are two scalars such that $x\,\vec{a} + y\,\vec{b} = 0$, then $x = 0,\ y = 0$.

(2) If $\vec{a},\ \vec{b}$ and $\vec{c}$ three non-zero coplanar vectors and *x*, *y* and z are three scalars such that

$x\vec{a} + y\vec{b} + z\vec{c} = 0$, then $x = 0,\ y = 0,\ z = 0$.

(3) If $\vec{a},\ \vec{b}$ be two given non collinear vectors, then every vector $\vec{r}$ can be expressed uniquely as a linear combination $\vec{r} = x\vec{a} + y\vec{b}; x, y$ being scalars.

(4) Similarly for three non–coplanar vectors, $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$

Test of collinearity.

(1) (i) **For two vectors** : $\vec{b} = L\vec{a}$, where *L* is a scalar.

(ii) **For three vectors:** $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$

(2) **Test of coplanarity:** (i) **For three vectors :** $\vec{a} = x\vec{b} + y\vec{c}$

(ii) **For four vectors :** $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$

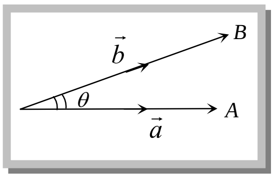
Where, $x + y + z + w = 0$

Scalar product (Dot product).

Scalar product : Scalar product of two vectors $\vec{a}$ and $\vec{b}$ is given by $|\vec{a}||\vec{b}|\cos\theta$, θ being angle between the vectors $\vec{a}$ and $\vec{b}$. Therefore, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Note : ❑ (i) Projection of $\vec{b}$ on $\vec{a}$ $= \hat{a}.\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$ (ii) Projection of $\vec{a}$ on $\vec{b}$ $= \vec{a}.\hat{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

(1) **Properties of scalar product**



(i) $\vec{a}.\vec{a} = |\vec{a}|^2 = a^2$ (ii) $\vec{a}.\vec{b} = \vec{b}.\vec{a}.$

(iii) $\vec{a}.(\vec{b} \pm \vec{c}) = \vec{a}.\vec{b} \pm \vec{a}.\vec{c}$ (iv) If *x* is any scalar, then $(x\,\vec{a}).\vec{b} = x(\vec{a}.\vec{b})$

(v) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = \vec{a}^2 - \vec{b}^2$ (vi) $(\vec{a}+\vec{b})^2 = \vec{a}^2 + 2\vec{a}.\vec{b} + \vec{b}^2$ (vii) $(\vec{a}-\vec{b})^2 = \vec{a}^2 - 2\vec{a}.\vec{b} + \vec{b}^2$

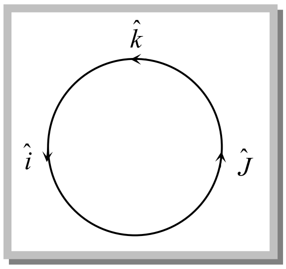
(2) **Angle between two vectors** : Let $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{j} + b_2\hat{j} + c_2\hat{k}.$ be two vectors inclined at an angle θ. then, $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $\Rightarrow \cos\theta = \frac{a_1, b_1 + a_2\,b_2 + a_3\,b_3}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$

(3) **Work done by a force** : If $\vec{F}$ is force and $\vec{d}$ is displacement, then,

Work done $\vec{F}.\vec{d} = Fd\cos\theta$, Where θ being angle between $\vec{F}$ and $\vec{d}$.

Vector product (Cross product).

The vector product of two vector $\vec{a}$ and $\vec{b}$ is defined by $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\;\hat{n}$



When θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to plane of $\vec{a}$, $\vec{b}$ and positive for a right handed rotation from $\vec{a}$ to $\vec{b}$.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$ then vector product

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties of vector product : (1) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (2) $\vec{a}\times\vec{a} = \vec{0}$ (3) $\vec{a}\times(\vec{b}\pm\vec{c}) = \vec{a}\times\vec{b} \pm \vec{a}\times\vec{c}$

(4) $(\vec{a}\times\vec{b})^2 = a^2b^2 - (\vec{a}\,.\,\vec{b})^2$ (5) $i\times i = j\times j = k\times k = 0$ (6) $i\times j = k,\; j\times k = i,\; k\times i = j$

(7) Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be two vectors then $\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & c_3 \end{vmatrix}$

(8) Two non-zero vector $\vec{a}$ and $\vec{b}$ are collinear iff, $\vec{a}\times\vec{b} = 0 \Rightarrow \vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{0}$

(9) (i) Vectors area of parallelogram whose adjacent sides are represented by vectors $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ is $|\vec{a}\times\vec{b}|$.

(ii) Area of triangle $= \frac{1}{2}|\vec{a}\times\vec{b}|$

(iii) Area of triangle ABC $= \frac{1}{2}|\bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{a}\times\bar{b}|$ where *a*, *b*, *c* are position vectors of *A*, *B*, *C* respectively.

(10) A vector of magnitude L_1 perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is given by $\pm\lambda\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$.

(11) Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

Note : ❑ If two vectors $\vec{a}$ and $\vec{b}$ are perpendicular then $\vec{a}.\vec{b}$ = 0 and if parallel, then $\vec{a}\times\vec{b}=0$.

Scalar triple product.

(1) **Scalar triple product** : Let $\vec{a},\ \vec{b},\ \vec{c}$ be any three vectors then the vector $\vec{a}.(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product of $\vec{a},\ \vec{b}$ and $\vec{c}$ The scalar triple product $(\vec{a}\times\vec{b}).\vec{c}$ is also denoted by the symbol $[\vec{a}\,\vec{b}\,\vec{c}\,]$.

Note : ❑ The scalar triple product is the volume of the parallelopiped whose adjacent sides are represented by vectors $\vec{a},\ \vec{b}$ and $\vec{c}$ is $\vec{c}$ i.e. volume of parallelopiped is $[\overline{a},\overline{b},\overline{c}]$.

(2) **Properties of scalar product :** (i) $[\hat{i}\,\hat{j}\,\hat{k}]=[\hat{j}\,\hat{k}\,\hat{i}]=[\hat{k}\,\hat{i}\,\hat{j}]=1$ (ii) $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$

(iii) Condition of coplanarity of three vectors $[\vec{a}\ \vec{b}\ \vec{c}]=0 \Rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$

(iv) Volume of tetrahedron ABCD is $\frac{1}{6}\left|\vec{AB}.(\vec{AC}\times\vec{AD})\right|$ (v) $[\vec{a}\,\vec{b}\,\vec{c}][\vec{a}'\vec{b}'\vec{c}']=\begin{bmatrix} \vec{a}.\vec{a}' & \vec{b}.\vec{a}' & \vec{c}.\vec{a}' \\ \vec{a}.\vec{b}' & \vec{b}.\vec{b}' & \vec{c}.\vec{b}' \\ \vec{a}.\vec{c}' & \vec{b}.\vec{c}' & \vec{c}.\vec{c}' \end{bmatrix}$

Reciprocal system of vectors.

If $\vec{a},\vec{b},\vec{c}$ are three non-zero non-collinear and non-coplaner vectors, then the three vectors $\vec{a},\vec{b},\vec{c}$ defined by $\vec{a}'=\frac{\overline{b}\times\vec{c}}{[\vec{a}\,\overline{b}\,\vec{c}]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\overline{b}\,\vec{c}]}$ are called reciprocal system of vectors for the vectors $\vec{a},\vec{b},\vec{c}$.

Vector triple product.

Let $\vec{a},\vec{b},\vec{c}$ be any three vector then the vector $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are called vector triple product of $\vec{a},\ \vec{b}$ and $\vec{c}$. $\vec{a}(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$ or $(\vec{a}\times\vec{b})\times\vec{c}=\{\vec{a}\times(\vec{b}\times\vec{c})\}=-(\vec{a}.\vec{c})\vec{b}+(\vec{a}.\vec{b})\vec{c}$

(1) If $\vec{a},\vec{b},\vec{c}$ are any three vectors, then, $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2$

(2) If $\vec{a},\vec{b},\vec{c}$ are non coplanar vectors then, $\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}$ are also non coplanar.

Scalar product of four vector.

The expression $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$ is called the scalar product of four vectors $\vec{a},\vec{b},\vec{c},\vec{d}$

$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{a}.\vec{d})(\vec{b}.\vec{c})$

Vector product of four vector.

The expression $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$ is called the vector product of four vectors $\vec{a},\vec{b},\vec{c},\vec{d}$

(1) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$. (Regard $\vec{a}\times\vec{b}$ as a single vector)

(2) $2.(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\vec{c}\vec{d}]\,\vec{b}-[\vec{b}\vec{c}\,\vec{d}]\vec{a}$. (Regard $\vec{c}\times\vec{d}$ as a single vector)

Rotation of a vector about an axis

Let $\vec{a}=(a_1,a_2,a_3)$. If the system is rotated about,

(1) *x*-axis through an angle α, then the new components of $\vec{a}$ are $(a_1, a_2\cos\alpha+a_3\sin\alpha, -a_2\sin\alpha+a_3\cos\alpha)$.

(2) *y*-axis through an angle α, then the new components of $\vec{a}$ are, $(-a_3\sin\alpha+a_1\cos\alpha, a_2, a_3\cos\alpha+a_1\sin\alpha)$.

(3) *z*-axis through an angle α, then the new components of $\vec{a}$ $(a_1\cos\alpha+a_2\sin\alpha, -a_1\sin\alpha+a_2\cos\alpha, a_3)$.

Equation of a plane in vector form

(1) (i) Equation of plane passing through the point with *P.V.* $\vec{a}$ and parallel to the plane containing $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+x\,\vec{b}+y\,\vec{c}$ or $[\vec{r}-\vec{a},\vec{b},\vec{c}]=0$.

(ii) The equation of a plane through three points $A(\vec{a}),B(\vec{b}),C(\vec{c})$ is $\vec{r}=(1-\lambda-\mu)\vec{a}+\lambda\,\vec{b}+\mu\,\vec{c}$ or $\vec{r}\,(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})=[\vec{a}\,\vec{b}\,\,\vec{c}]$. This shows that normal of this plane is $\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$.

(iii) Equation of a plane which is at a distance *p* from the origin and having a unit normal $\vec{n}$ is $\vec{r}\,.\,\vec{n}=p$.

(iv) Equation of a plane through $A(\vec{a})$ and having a unit normal $\vec{n}$ is $(\vec{r}-\vec{a}).\,\vec{n}=0$.

(v) Length of perpendicular from the origin upon the plane passing through $A(\vec{a}),B(\vec{b}),C(\vec{c})$ is

$$\frac{|\,[\vec{a}\;\vec{b}\;\,\vec{c}]|}{|\,\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\,|}.$$

(2) **Equation of line in vector form:**

(i) Vector equation of a line passing through a point $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r}=\vec{a}+t\,\vec{b}$, where t is an orbitary constant.

(ii) Vector equation of a straight line passing through two points $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$.

Some important points

(1) Components of a vector $\vec{r}$, in the direction of a is $\frac{\vec{r}\,.\,\vec{a}}{|\,\vec{a}|^2}\vec{a}$, and perpendicular to $\vec{a}$ is $\vec{r}-\frac{\vec{r}\,.\,\vec{a}}{|\,\vec{a}|^2}\vec{a}$.

(2) The vector moment or torque *M* of a force $\vec{F}$ acting at *A*, about the point *O* is given by $\vec{M}=\vec{r}\times\vec{F}=\vec{OA}\times\vec{F}$.

(3) $\vec{O}$ is parallel to every vector.

(4) Three vectors $\vec{a},\vec{b},\vec{c}$ form a right handed and left handed system according as $[\vec{a}\,\vec{b}\,\vec{c}]>0$ or <0 respectively. (5) $(\vec{a}\,.\,\vec{b})^2+(\vec{a}\times\vec{b})^2=a^2b^2$.

(6) $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ and $|\vec{a}-\vec{b}| \ge \left| |\vec{a}|-|\vec{b}| \right|$ also $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ hold iff $\vec{a}$ and $\vec{b}$ have the same direction.

(7) If $\vec{a}, \vec{b}, \vec{c}$ are three vectors of the points A, B, C respectively, then the perpendicular distance from C to the line AB is $\frac{|(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})|}{|\vec{b}-\vec{a}|}$.

Some useful results

(1) If $\vec{a}$ and $\vec{b}$ are two vectors then $(\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$.

(2) If $\vec{a}$ is the any vector then $(\vec{a}\times\vec{i})^2+(\vec{a}\times\vec{j})^2+(\vec{a}\times\vec{k})^2 = 2|\vec{a}|^2$.

(3) $\vec{i}\times(\vec{a}\times\vec{i})+\vec{j}\times(\vec{a}\times\vec{j})+\vec{k}\times(\vec{a}\times\vec{k}) = 2\vec{a}$.

(4) $[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

(5) $[\vec{a}-\vec{b},\ \vec{b}-\vec{c},\ \vec{c}-\vec{a}] = 0$.

(6) $[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$.

(7) If $\vec{a}, \vec{b}\, \vec{c}$ are coplanar, then so are $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ and $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$.

Also clearly $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ are coplanar.

(8) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, If any two of the three vectors $\bar{a}, \bar{b}, \bar{c}$ are collinear or equal. (9) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, If $\vec{a}, \vec{b}, \vec{c}$ are collinear.

(10) Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ will be coplanar If $[\vec{d}\ \vec{b}\ \vec{c}]+[\vec{d}\ \vec{c}\ \vec{a}]+[\vec{d}\ \vec{a}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]$.

(11) Let $\vec{p} = x_1\vec{a}+x_2\vec{b}+x_3\vec{c}, \vec{q} = y_1\vec{a}+y_2\vec{b}+y_3\vec{c}, \vec{r} = z_1\vec{a}+z_2\vec{b}+z_3\vec{c}$, then $[\vec{p}\ \vec{q}\ \vec{r}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}[abc]$.

(12) If $\vec{a}\ \vec{b}\ \vec{c}$ are three vectors such that $\vec{a}, \vec{b}, \vec{c}$ are perpendicular to $\vec{b}+\vec{c},\ \vec{c}+\vec{a},\ \vec{a}+\vec{b}$ respectively, then $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2}$.