# Differentiation

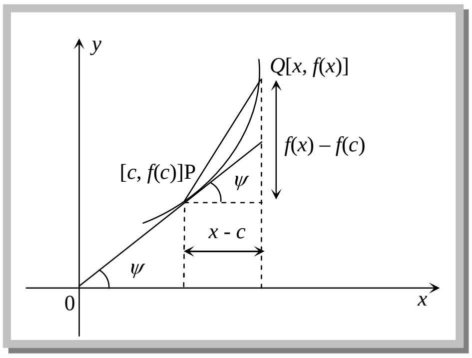
## Introduction

The rate of change of one quantity with respect to some another quantity has a great importance. For example, the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity is called its acceleration.

The rate of change of a quantity '*y*' with respect to another quantity '*x*' is called the derivative or differential coefficient of *y* with respect to *x*.

## Geometrical meaning of derivatives

Consider the curve $y = f(x)$. Let $f(x)$ be differentiable at $x = c$. Let $P(c, f(c))$ be a point on the curve and $Q(x, f(x))$ be a neighbouring point on the curve. Then,




Slope of the chord $PQ = \frac{f(x) - f(c)}{x - c}$. Taking limit as $Q \to P$, i.e., $x \to c$,

we get $\lim_{Q \to P} (\text{slope of the chord } PQ) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ ......(i)

As $Q \to P$, chord *PQ* becomes tangent at *P*.

Therefore from (i), we have

Slope of the tangent at $P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left(\frac{df(x)}{dx}\right)_{x = c}$.

**Note** : ❑ Thus, the derivatives of a function at a point $x = c$ is the slope of the tangent to curve, $y = f(x)$ at point $(c, f(c))$.

## Derivatives of standard function

(1) $\frac{d}{dx}(c) = 0$

(2) $\frac{d}{dx}(ax) = a$

(3) $\frac{d}{dx}(x^n) = n x^{n-1}$

(4) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(5) $\frac{d}{dx}(e^x) = e^x$

(6) $\frac{d}{dx}(a^x) = a^x \log_e a$

(7) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

(8) $\frac{d}{dx}(\sin x) = \cos x$

(9) $\frac{d}{dx}(\cos x) = -\sin x$

(10) $\frac{d}{dx}(\tan x) = \sec^2 x$

(11) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(12) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(13) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(14) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}, -1 < x < 1$

(15) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}, -1 < x < 1$

(16) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}, -\infty < x < \infty$

(17) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$

(18) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}, |x| > 1$

(19) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}, |x| > 1$

## Theorems on differentiation

(1) $\frac{d}{dx}[kf(x)] = k \frac{d}{dx}[f(x)]$, where *k* is a constant

(2) $\frac{d}{dx}[f_1(x) \pm f_2(x) \pm f_3(x) \pm ---] = \frac{d}{dx}[f_1(x)] \pm \frac{d}{dx}[f_2(x)] \pm \frac{d}{dx}[f_3(x)] \pm ------$

(4) $\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}g(x) + g(x).\frac{d}{dx}f(x)$

(5) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

## Chain rule

If $y = f(u), u = g(x),$ then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ and $y = f(u), u = g(v), v = h(x),$ then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

## Logarithmic differentiation

To find the derivative of $[f(x)]^{g(x)}$, where $f(x)$ and $g(x)$, are the function of $x$ .

We proceed as follows: Let, $y = [f(x)]^{g(x)}$

Taking log of both sides, $\log y = g(x).\log\{f(x)\}$

By applying product rule, $\frac{dy}{dx} = [f(x)]^{g(x)}\left[\frac{g(x)}{f(x)}.\frac{df(x)}{dx} + \log\{f(x)\}.\frac{dg(x)}{dx}\right]$.

## Differentiation of parametric equations

Sometimes $x$ and $y$ are given as a function of single variable, $x = \phi(t),\ y = \psi(t)$, Then, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\psi'(t)}{\phi'(t)}$

## Differentiation of implicit functions

If the variables $x$ and $y$ are connected by a relation of the form $f(x, y) = 0$ and it is not possible or convenient to express $y$ as a function $x$ in the form $y = \phi(x)$, then $y$ is said to be an implicit function of $x$. To find $\frac{dy}{dx}$ in such a case, we differentiate both sides of the given relation with respect to $x$, keeping in mind that the derivative of $\phi(y)$ w.r.t. $x$ is $\frac{d\phi}{dy}.\frac{dy}{dx}$. For example: $x^3 + y^3 - 3axy = 0$

Differentiating w.r.t. $x$, $3x^2 + 3y^2\frac{dy}{dx} - 3a(1.y + x.\frac{dy}{dx}) = 0 \Rightarrow (x^2 - ay) + (y^2 - ax)\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{(x^2 - ay)}{(y^2 - ax)}$.

Note : ❑ We can find $\frac{dy}{dx}$ by this formula also $\frac{dy}{dx} = -\frac{f(x,y)_x}{f(x,y)_y} = -\frac{\text{Differentiation w.r.t. } x}{\text{Differentiation w.r.t. } y}$

## Differentiation of function with respect to another function

In this section we will discuss derivative of a function with respect to another function. Let $u = f(x)$ and $v = g(x)$ be two functions of $x$. Then, to find the derivative of $f(x)$ w.r.t. $g(x)$ i.e., to find $\frac{du}{dv}$ we use the following formula $\frac{du}{dv} = \frac{du/dx}{dv/dx}$

Thus, to find the derivative of $f(x)$ w.r.t. $g(x)$, we first differentiate both w.r.t. $x$ and then divide the derivative of $f(x)$ w.r.t. $x$ by the derivative of $g(x)$ w.r.t. $x$.

## Differentiation by trigonometrical substitution

For trigonometrical substitutions following formulae and substitution should be remembered

**Formulae**

(1) $\sin^{-1}x+\cos^{-1}x=\pi/2$

(2) $\tan^{-1}x+\cot^{-1}x=\pi/2$

(3) $\sec^{-1}x+\operatorname{cosec}^{-1}x=\pi/2$

(4) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$

(5) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left[xy\mp\sqrt{(1-x^2)(1-y^2)}\right]$

(6) $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\left[\frac{x\pm y}{1\mp xy}\right]$

(7) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

(8) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$

(9) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(10) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$

(11) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

(12) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

(13) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(14) $\sin^{-1}(-x)=-\sin^{-1}x$

(15) $\cos^{-1}(-x)=\pi-\cos^{-1}x$

(16) $\tan^{-1}(-x)=-\tan^{-1}x$ or $\pi-\tan^{-1}x$

(17) $\frac{\pi}{4}-\tan^{-1}x=\tan^{-1}\left(\frac{1-x}{1+x}\right)$

**Some suitable substitutions**

| S. N. | Function | Substitution | S. N. | Function | Substitution |
|---|---|---|---|---|---|
| (1) | $\sqrt{a^2-x^2}$ | $x=a\sin\theta$ or $a\cos\theta$ | (2) | $\sqrt{x^2+a^2}$ | $x=a\tan\theta$ or $a\cot\theta$ |
| (3) | $\sqrt{x^2-a^2}$ | $x=a\sec\theta$ or $a\operatorname{cosec}\theta$ | (4) | $\sqrt{\frac{a-x}{a+x}}$ | $x=a\cos 2\theta$ |
| (5) | $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ | $x^2=a^2\cos 2\theta$ | (6) | $\sqrt{ax-x^2}$ | $x=a\sin^2\theta$ |
| (7) | $\sqrt{\frac{x}{a+x}}$ | $x=a\tan^2\theta$ | (8) | $\sqrt{\frac{x}{a-x}}$ | $x=a\sin^2\theta$ |
| (9) | $\sqrt{(x-a)(x-b)}$ | $x=a\sec^2\theta-b\tan^2\theta$ | (10) | $\sqrt{(x-a)(b-x)}$ | $x=a\cos^2\theta+b\sin^2\theta$ |

## Differentiation by inverse function

To find out the differentiation of such functions i.e., $\sin^{-1}[f(x)]$, we put a term in place of $f(x)$, such that, $f(x)$ become, $\sin\theta$, Now the form of the function $\sin^{-1}(\sin\theta)=\theta$ and differentiation of $\theta$ can be obtained.

## Differentiation of infinite series

When $y$ is given, in the form of infinite series of $x$, then we differentiate the function in following way:

(1) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+.....}}}$

Then $y=\sqrt{f(x)+y}$, $\therefore \frac{dy}{dx}=\frac{f'(x)}{2y-1}$.

(2) If $y=f(x)^{f(x)^{f(x)^{----\infty}}}$, then $y=[f(x)]^y$

$\therefore \log y=y\log[f(x)]\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{y.f'(x)}{f(x)}+\log f(x).\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y \log f(x)]}$$

(3) If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + .....\infty}}$ then $\frac{dy}{dx} = \frac{yf'(x)}{2y - f(x)}$

## Successive differentiation

(1) **Definition and notation :** If $y$ is a function of $x$ and is differentiable with respect to $x$, then its derivative $\frac{dy}{dx}$ can be found which is known as derivative of first order. If the first derivative $\frac{dy}{dx}$ is also a differentiable. function, then it can be further differentiate with respect to $x$ and this derivative is denoted by $d^2y/dx^2$ which is called the second derivative of $y$ with respect to $x$ further if $\frac{d^2y}{dx^2}$ is also differentiable then its derivative is called third derivative of $y$ which is denoted by $\frac{d^3y}{dx^3}$. Similarly $n^{th}$ derivative of $y$ is denoted by $\frac{d^ny}{dx^n}$. All these derivatives are called as successive derivative and this process is known as successive differentiation. We also use the following symbols for the successive derivatives of $y = f(x)$ :

$y_1,\ y_2,\ y_3, ........., y_n, ......$ $\qquad$ $y^I,\ y^{II},\ y^{III} ........., y^n, ......$

$Dy,\ D^2y,\ D^3y ........., D^ny, ......$ (where $D = \frac{d}{dx}$) $\qquad$ $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ......., \frac{d^ny}{dx^n}, .........\ ..$

$f^I(x),\ f^{II}(x),\ f^{III}(x), ........ ., f^n(x), ......$

If $y = f(x)$, then the value of the $n^{th}$ order derivative at $x = a$ is usually denoted by

$\left(\frac{d^ny}{dx^n}\right)_{x=a}$ or $(y_n)_{x=a}$ or $(y^n)_{x=a}$ or $f^n(a)$

(2) $n^{th}$ Derivatives of some standard functions –

(i) (a) $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$ $\qquad$ (b) $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$, where $m > n$

**Particular cases :**

(i) When $m = n$

$D^n\{(ax+b)^n\} = a^n.n!$

(ii) When $a = 1, b = 0$, then $y = x^m$

$\therefore D^n(x^m) = m(m-1)......... .(m-n+1)x^{m-n} = \frac{m!}{(m-n)!}x^{m-n}$

(iii) When $a = 1$, $b = 0$ and $m = n$, then $y = x^n$

$\therefore D^n(x^n) = n!$

(iv) When $m = -1$, $y = \frac{1}{(ax+b)}$

$y = a^n(-1)(-2)(-3)........(-n)(ax+b)^{-1-n}$

$= a^n(-1)^n(1.2.3......n)(ax+b)^{-1-n} = \frac{a^n(-1)^n n!}{(ax+b)^{n+1}}$

(3) $\frac{d^n}{dx^n}\log(ax+b) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$ $\qquad$ (4) $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(5) $\frac{d^n(a^x)}{dx^n} = a^x \log a$

(6) (i) $\frac{d^n}{dx^n} e^{ax} \sin(bx+c) = r^n e^{ax} \sin(bx+c+n\phi)$

where $r = \sqrt{a^2+b^2}$; $\phi = \tan^{-1}\frac{b}{a}$, $y = e^{ax}\sin(bx+c)$

(ii) $\frac{d^n}{dx^n} e^{ax} \cos(bx+c) = r^n e^{ax} \cos(bx+c+n\phi)$

## Leibnitz's theorem

If $u$ and $v$ are two functions of $x$ such that their $n$th derivative exist, then

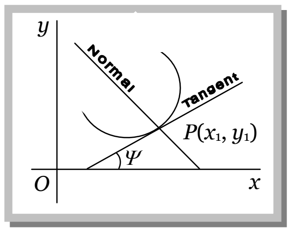
$$D^n(uv) = {}^nC_0(D^n u)v + {}^nC_1 D^{n-1}u.Dv + {}^nC_2 D^{n-2}u.D^2v + ....... + {}^nC_r D^{n-r}u.D^rv + ...... + {}^nC_n u.D^n v$$

**Note :** ❑ The success in finding the $n$th derivative by this theorem lies in the proper selection of first and second function. Here first function should be selected whose $n$th derivative can be found by standard formulae. Second function should be such that on successive differentiation, at some stage, it becomes zero so that we need not to write further terms.

# Application of Derivatives

## Tangent and Normal

If tangent is drawn on the curve $y = f(x)$ at point $P$, then the gradiant of this tangent at point $P$ will be equal to derivative at that point. Hence, if tangent of curve $y = f(x)$ at point on curve makes an angle $\psi$ with positive $x$-direction, then, $\left(\frac{dy}{dx}\right)_P = \tan\psi$ = Slope of the tangent




If tangent is parallel to $x$-axis, $\psi = 0 \Rightarrow \left(\frac{dy}{dx}\right)_P = 0$

If tangent is perpendicular to $x$-axis, $\psi = \frac{\pi}{2} \Rightarrow \left(\frac{dy}{dx}\right)_P = \infty$ or $\left(\frac{dx}{dy}\right)_P = 0$

and Slope of the normal at $P = \frac{-1}{\text{Slope of tangent at } P} = \frac{-1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$.

## Equation of tangent

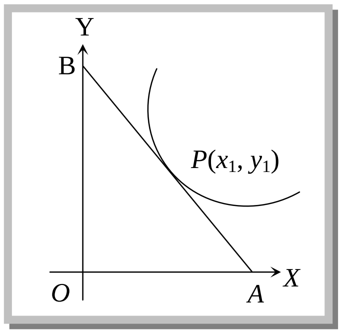
(1) **Tangent and Normal :** The equation of tangent of curve $y = f(x)$ at point $P(x_1, y_1)$, $y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$

**Note :** ❑ If the tangent at $P(x_1, y_1)$ of the curve $y = f(x)$ is parallel to the $x$-axis (or perpendicular to $y$-axis) then $\Psi = 0$ *i.e.* its slope will be zero $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$. The converse is also true. Hence the tangent at $(x_1, y_1)$ is parallel to $x$-axis $\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

❑ If the tangent at $P(x_1, y_1)$ of the curve $y = f(x)$ is parallel to *y*-axis (or perpendicular to *x*-axis), then $\Psi = \frac{\pi}{2}$ and its slope will be infinity *i.e.* $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$. The converse is also true. Hence the tangent at $(x_1, y_1)$ is parallel to *y*-axis $\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$.

❑ If at any point $P(x_1, y_1)$ on the curve $y = f(x)$, the tangent makes equal angles with the axes, then at the point *P*, $\Psi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Hence at *P*, $\tan \Psi = \frac{dy}{dx} = \pm 1$. The converse of the result is also true, thus at $(x_1, y_1)$ the tangent line makes equal angles with the axes $\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm 1$.

(2) **Length of intercepts made on axes by the tangent :** Equation of tangent at any point $(x_1, y_1)$ to the curve $y = f(x)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$ ......(i) Equation of *x*-axis, *y* = 0 .....(ii)




Equation of *y*-axis, *x* = 0 ......(iii)

Solving (i) and (ii), we get, $x = x_1 - \left\{\frac{y_1}{(dy/dx)_{x_1, y_1}}\right\}$,

$\therefore x - \text{intercept} = OA = x_1 - \left\{\frac{y_1}{(dy/dx)_{(x_1, y_1)}}\right\}$

similarly solving (i) and (iii), we get, *y*-intercept $= OB = y_1 - x_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.

(3) **Length of perpendicular from origin to the tangent :** The length of perpendicular from origin (0, 0) to the tangent drawn at the point $(x_1, y_1)$ of the curve $y = f(x)$.

$$p = \left|\frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

## Equation of normal

The equation of the normal of curve $y = f(x)$ at point $P(x_1, y_1) = \frac{-1}{(dy/dx)_P}(x - x_1)$ or $y - y_1 = -\left(\frac{dx}{dy}\right)_P (x - x_1)$

**Some facts about the normal :**

(1) If normal makes an angle of $\theta$ with positive direction of *x*-axis then, $-\left(\frac{dx}{dy}\right) = \tan\theta$ or $\frac{dy}{dx} = -\cot\theta$

(2) If normal is parallel to *x*-axis then, $-\left(\frac{dx}{dy}\right) = 0$ or $\frac{dy}{dx} = \infty$

(3) If normal is parallel to *y*-axis then, $-\left(\frac{dx}{dy}\right) = \infty$ or $\frac{dy}{dx} = 0$

(4) If normal is equally inclined from both the axes or cuts equal intercept then, $-\left(\frac{dx}{dy}\right)=\pm 1$ or $\left(\frac{dy}{dx}\right)=\pm 1$

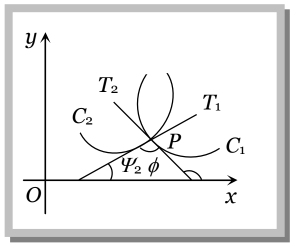
(5) The length of perpendicular from origin to normal is $p'=\left|\frac{x_1+y_1\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|$

(6) The length of intercept made by normal on $x$-axis $=x_1+y_1\left(\frac{dy}{dx}\right)$ and

length of intercept on $y$-axis $=y_1+x_1\left(\frac{dx}{dy}\right)$

## Angle of intersection of two curves

The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection. Let $C_1$ and $C_2$ be two curves having equations $y=f(x)$ and $y=g(x)$ respectively. Let $PT_1$ and $PT_2$ be tangents to the curves $C_1$ and $C_2$ respectively at their common point of intersection. Then the angle $\phi$ between $PT_1$ and $PT_2$ is the angle of intersection of $C_1$ and $C_2$. Let $\psi_1$ and $\psi_2$ be the angles made by $PT_1$ and $PT_2$ with the positive direction of $x$-axis in anticlockwise sense. Then $m_1=\tan\psi_1$ = slope of the tangent to $y=f(x)$ at $P=\left(\frac{dy}{dx}\right)_{C_1}$ and, $m_2=\tan\psi_2=$ slope of the tangent to $y=g(x)$ at $P=\left(\frac{dy}{dx}\right)_{C_2}$




From figure, it follows that, $\phi=\psi_1-\psi_2$

$\Rightarrow \tan\phi=\tan(\psi_1-\psi_2)$ $\quad\Rightarrow \tan\phi=\frac{\tan\psi_1-\tan\psi_2}{1+\tan\psi_1\tan\psi_2}$ $\quad\Rightarrow \tan\phi=\frac{(dy/dx)_{C_1}-(dy/dx)_{C_2}}{1+(dy/dx)_{C_1}(dy/dx)_{C_2}}$

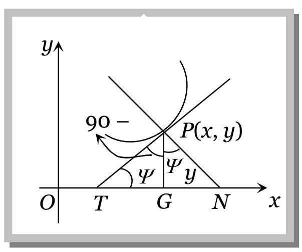
The other angle between the tangents is $180^o-\phi$. Generally the smaller of these two angles is taken to be the angle of intersection.

**Note** : ❑ If two curves intersect orthogonally *i.e.* at right angle then $\phi=\frac{\pi}{2}$ so the condition will be $\left(\frac{dy}{dx}\right)_{c_1}\cdot\left(\frac{dy}{dx}\right)_{c_2}=-1$.

## Lengths of tangent, normal, subtangent and subnormal

Let the tangent and normal at a point $P(x,y)$ on the curve $y=f(x)$, meet the $x$-axis at $T$ and $N$ respectively. If $G$ is the foot of the ordinate at $P$, then $TG$ and $GN$ are called the cartesian subtangent and subnormal, while the lengths $PT$ and $PN$ are called the lengths of the tangent and normal respectively.

If $PT$ makes angle $\psi$ with $x$-axis, then $\tan\psi=\frac{dy}{dx}$. From figure, we find that



Subtangent $TG = y\cot\psi = \dfrac{y}{\left(\dfrac{dy}{dx}\right)}$, Subnormal $= GN = y\tan\psi = y\dfrac{dy}{dx}$

Length of the tangent $= PT = y\,\text{cosec}\,\psi = y\sqrt{1+\cot^2\psi} = \dfrac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}}$

Length of the normal $= PN = y\sec\psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$.

## Rolle's theorem

**Statement** – Let $f$ be a real valued function defined on the closed interval $[a, b]$ such that

(1) It is continuous on the closed interval $[a, b]$. (2) It is differentiable on the open interval $(a, b)$ and

(3) $f(a) = f(b)$

Then there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.

## Lagrange's mean value theorem

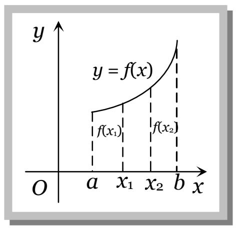
**Statement** – Let $f(x)$ be a function defined on $[a, b]$ such that

(1) It is continuous on $[a, b]$. (2) It is differentiable on $(a, b)$.

Then there exist a real number $c \in (a, b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

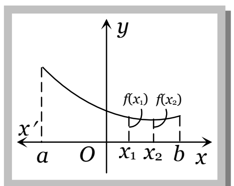
## Increasing and Decreasing functions

(1) **Strictly increasing function :** A function $f(x)$ is said to be a strictly increasing function on $(a, b)$, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.




Thus, $f(x)$ is strictly increasing on $(a, b)$, if the values of $f(x)$ increase with the increase in the values of $x$.

(2) **Strictly decreasing function :** A function $f(x)$ is said to be a strictly decreasing function on $(a, b)$, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$. Thus, $f(x)$ is strictly decreasing on $(a, b)$, if the values of $f(x)$ decrease with the increase in the values of $x$.

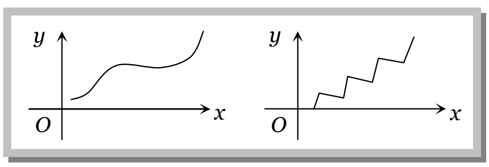



(3) **Monotonic function :** A function $f(x)$ is said to be monotonic on an interval $(a, b)$ if it is either increasing or decreasing on $(a, b)$.

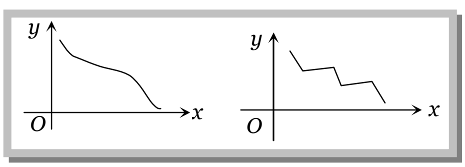
(i) Monotonic increasing function : A function is said to be a monotonic increasing function in defined interval if, $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$

or $x_1 > x_2 \Rightarrow f(x_1) \nless f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) \ngtr f(x_2)$ or $x_1 < x_2 \Rightarrow f(x_1) \ngtr f(x_2)$




(ii) Monotonic decreasing function: A function is said to be a monotonic decreasing function in defined interval, if $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$

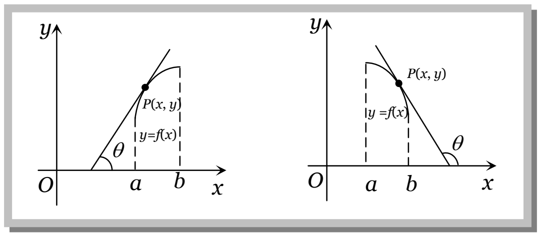


or $x_1 > x_2 \Rightarrow f(x_1) \ngtr f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ or $x_1 < x_2 \Rightarrow f(x_1) \nless f(x_2)$

(4) **Necessary and sufficient conditions for monotonic functions :** In this section we intend to see how we can use derivative of a function to determine where it is increasing and where it is decreasing

(i) Necessary condition : From figure we observe that if $f(x)$ is an increasing function on $(a, b)$, then tangent at every point on the curve $y = f(x)$ makes an acute angle $\theta$ with the positive direction of $x$-axis.




$\therefore \tan\theta > 0 \Rightarrow \frac{dy}{dx} > 0$ or $f'(x) > 0$ for all $x \in (a, b)$

It is evident from figure that if $f(x)$ is a decreasing function on $(a, b)$, then tangent at every point on the curve $y = f(x)$ makes an obtuse angle $\theta$ with the positive direction of $x$-axis.

$\therefore \tan\theta < 0 \Rightarrow \frac{dy}{dx} < 0$ or $f'(x) < 0$ for all $x \in (a, b)$.

Thus, $f'(x) > 0 (< 0)$ for all $x \in (a, b)$ is the necessary condition for a function $f(x)$ to be increasing (decreasing) on a given interval $(a, b)$. In other words, if it is given that $f(x)$ is increasing (decreasing) on $(a, b)$, then we can say that $f'(x) > 0 (< 0)$.

(ii) Sufficient condition : Theorem : Let $f$ be a differentiable real function defined on an open interval $(a, b)$.

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on $(a, b)$.

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on $(a, b)$.

**Corollary** – Let $f(x)$ be a function defined on $(a, b)$.

(a) If $f'(x) > 0$ for all $x \in (a, b)$, except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is increasing on $(a, b)$.

(b) If $f'(x) < 0$ for all $x \in (a, b)$, except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is decreasing on $(a, b)$.

(5) **Test for monotonicity :** (i) At a point : (a) Function $f(x)$ will be monotonic increasing in domain at a point if and only if, $f'(a) > 0$

(b) Function $f(x)$ will be monotonic decreasing in domain at a point if and only if, $f'(a) < 0$.

(ii) In an interval : Function $f(x)$, defined in $[a, b]$ is

(a) Monotonic increasing in $(a, b)$ if, $f'(x) \geq 0$, $a < x < b$

(b) Monotonic increasing in $[a, b]$ if, $f'(x) \geq 0$, $a \leq x \leq b$

(c) Strictly increasing in $[a, b]$, if, $f'(x) > 0$, $a \leq x \leq b$

(d) Monotonic decreasing in $(a, b)$, if, $f'(x) \leq 0$, $a < x < b$

(e) Monotonic decreasing in $[a, b]$, if, $f'(x) \leq 0$, $a \leq x \leq b$

(f) Strictly decreasing in $[a, b]$, if, $f'(x) < 0$, $a \leq x \leq b$

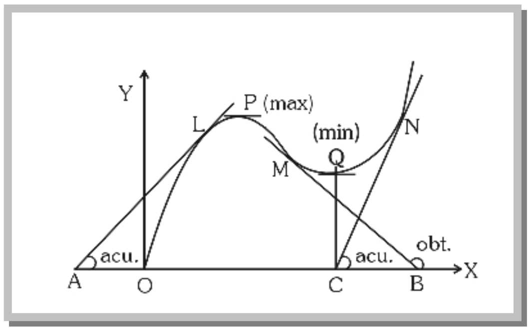
## Maxima and Minima

(1) **Working rule for Maxima or Minima :** From the figure it is clear that at $P$ the function $y = f(x)$ is maximum and at $Q$ it is minimum. At both these points tangent is parallel to $x$-axis so that its slope is zero.

$\therefore \frac{dy}{dx} = 0$ or $f'(x) = 0$ for both Maxima and Minima.

**Above is the condition for Maxima and Minima.**

(2) **Criteria for Maxima and Minima :** Let $x = a, b, c$ be the values of *x* given by $\frac{dy}{dx} = 0$. Consider the point $x = a$ i.e., *P* where *y* is Maximum. From the figure it is clear that tangent at any point $x = a - h$ will make an acute angle with *x*-axis just as at *L*, and tangent at $x = a + h$ will make an obtuse angle just as at *M*. Thus,




For $x = a$, $\frac{dy}{dx} = 0$,

For *x* slightly < *a*, $\frac{dy}{dx} = +ive$, For *x* slightly > *a*, $\frac{dy}{dx} = -ive$

Hence, if *y* is max. at *x* = *a*, then $\frac{dy}{dx}$ changes sign from +*ive* to –*ive* for values of *x* less than *a* and greater than *a* in that order. Now consider the point $x = b$ i.e., *Q* where *y* is **Minimum.**

From the figure it is clear that tangent at any point *x* = *b* – *h* will make an obtuse angle with *x*-axis just as at *M* and tangent at *x* = *b* + *h* will make an acute angle just as at *N*. Thus,

For $x = b$, $dy/dx = 0$

For *x* slightly < *b*, $dy/dx = -ive$ (Obtuse)

For *x* slightly > *b*, $dy/dx = +ive$ (Acute)

Hence if *y* is minimum at $x = b$, then $dy/dx$ changes sign from –*ive* to +*ive* for values of *x* less than *b* and greater than *b* in that order.

**Working rule :** Calculate $dy/dx = 0$ and solve for *x* and say $x = a, b, c$ etc.

Put values of *x* slightly less than *a* in $dy/dx$ and values of *x* slightly greater than *a*.

If $\frac{dy}{dx}$ changes sign from +*ive* to –*ive*, then Maxima at $x = a$.

If $\frac{dy}{dx}$ changes sign from –*ive* to +*ive*, then Minima at $x = a$.

In case there is no change of sign then neither a maximum nor a minimum.

**Second method :** Calculate *dy*/*dx* = 0 and solve for *x*. Suppose one root of *dy*/*dx* = 0 is at *x* = *a*.

If $\frac{d^2y}{dx^2} = -ive$ for $x = a$, then Maxima at $x = a$.

If $\frac{d^2y}{dx^2} = +ive$ for *x* = *a*, then Minima at *x* = *a*.

If $\frac{d^2y}{dx^2} = 0$ at *x* = *a*, then find $\frac{d^3y}{dx^3}$. If $\frac{d^3y}{dx^3} \neq 0$ at *x* = *a*, then neither maximum nor minimum at $x = a$. If $\frac{d^3y}{dx^3} = 0$ at $x = a$, then find $\frac{d^4y}{dx^4}$. If $\frac{d^4y}{dx^4} > 0$ i.e., +*ive* at $x = a$, then *y* is minimum at $x = a$ and if $\frac{d^4y}{dx^4} < 0$ i.e., –*ive* at $x = a$, then *y* is max. at $x = a$ and so on.

**Generalisation :** If $f'(a) = 0 = f''(a) = \text{.........} \ \text{........} = f^{n-1}(a)$. But $f^n(a) \neq 0$, then,

(i) *x* = *a* is the point of Maxima for $f(x)$, if $f^n(a) < 0$ and *n* is even.

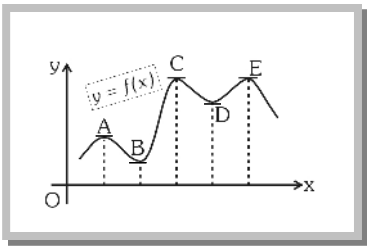
(ii) *x* = *a* is the point of Minima for *f*(*x*) if $f^n(a) > 0$ and *n* is even.

(iii) $x = a$ will be neither point of Maxima nor point of Minima if $n$ is odd.

(3) **Local Maxima and Local Minima :** In the previous section we have discussed about the greatest (maximum) and the least (minimum) values of a function in its domain. But there may be points in the domain of a function where the function does not attain the greatest (or the least) value but the values at these points are greater than or less than the values of the function at the neighbouring points. Such points are known as the points of local minimum or local maximum and we will be mainly discussing about the local maximum and local minimum values of a function.

(i) **Local maximum :** A function $f(x)$ is said to attain a local maximum at $x = a$ if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that, $f(x) < f(a)$ for all $x \in (a-\delta, a+\delta)$, $x \neq a$ or $f(x) - f(a) < 0$ for all $x \in (a-\delta,\ a+\delta), x \neq a$

In such a case $f(a)$ is called the local maximum value of $f(x)$ at $x = a$.




(ii) **Local minimum :** A function $f(x)$ is said to attain a local minimum at $x = a$, if there exists a neighbourhood $(a-\delta,\ a+\delta)$ of $a$ such that, $f(x) > f(a)$ for all $x \in (a-\delta,\ a+\delta),\ x \neq a$ or $f(x) - f(a) > 0$ for all $x \in (a-\delta, a+\delta), x \neq a.$

The value of the function at $x = a$ i.e., $f(a)$ is called the local minimum value of $f(x)$ at $x = a$.

The points at which a function attains either the local maximum values or local minimum values are known as the extreme points or turning points and both local maximum and local minimum values are called the extreme values of $f(x)$. Thus a function attains an extreme value at $x = a$, if $f(a)$ is either a local maximum value or a local minimum value.

**Note** : ❑ The maximum and minimum points are also known as extreme points.

❑ A function may have more than one maximum and minimum points.

❑ A maximum value of a function $f(x)$ in an interval $[a, b]$ is not necessarily its greatest value in that interval. Similarly, a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.

❑ If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.

❑ Monotonic functions do not have extreme points.

(4) **Application in mechanics :** If any particle moves with given speed $s$, then

$$\text{Velocity} = v = \frac{ds}{dt}, \text{ Acceleration} = a = \frac{d^2s}{dt^2} = \frac{dv}{dt}.$$