Exponential and Logarithmic Series

Exponential series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\ ....\ \infty$$

$e = 2.71$ (approx.) e is an irrational number.

Exponential theorem

Let $a > 0$, then for all real values of x

$$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + ...\ ....\infty$$

Some important results from exponential series

We have the exponential series

(1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ...\ \infty = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ......(i)

(2) Replacing x by $-x$, in (i), we obtain $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ...\infty = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ ......(ii)

(3) Putting $x = 1$ in (i) and (ii), we get,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...\infty = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(4) From (i) and (ii), we obtain

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...\infty \text{ or } \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

(5) $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + ...\infty = \sum_{n=0}^{\infty} \frac{1}{(2n)!}$

$$\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + ...\infty = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

Some standard results

(1) $\sum_{n=0}^{\infty} \frac{1}{n!} = e = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{(n-k)!} = e$

(2) $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{........} \infty = e - 1$

(3) $\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{........} \infty = e - 2$

(4) $\sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{........} \infty = e - 1$

(5) $\sum_{n=0}^{\infty} \frac{1}{(n+2)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{........} \infty = e - 2$

(6) $\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{........} \infty = e - 2$

(7) $\sum_{n=0}^{\infty} \frac{1}{(2n)!} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \text{........} = \frac{e + e^{-1}}{2} = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!}$

(8) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \text{........} = \frac{e - e^{-1}}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

(9) $e^{ax} = 1 + \frac{(ax)}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \text{........} + \frac{(ax)^n}{n!} + \text{......} \infty$

$\therefore T_{n+1}$ = General term in the expansion of $e^{ax} = \frac{(ax)^n}{n!}$ and Coefficient of x^n in $e^{ax} = \frac{a^n}{n!}$

(10) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{.....} + \frac{x^n}{n!} + \text{......} \infty$

$\therefore T_{n+1}$ = General term in the expansion of $e^x = \frac{x^n}{n!}$ and Coefficient of x^n in $e^x = \frac{1}{n!}$

(11) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \text{.......} + (-1)^n \frac{x^n}{n!} + \text{.....} \infty$

$\therefore T_{n+1}$ = General term in the expansion of $e^{-x} = (-1)^n \frac{x^n}{n!}$ and Coefficient of x^n in $e^{-x} = \frac{(-1)^n}{n!}$

(12) $\sum_{n=0}^{\infty} \frac{n}{n!} = e = \sum_{n=1}^{\infty} \frac{n}{n!}$

(13) $\sum_{n=0}^{\infty} \frac{n^2}{n!} = 2e = \sum_{n=1}^{\infty} \frac{n^2}{n!}$

(14) $\sum_{n=0}^{\infty} \frac{n^3}{n!} = 5e = \sum_{n=1}^{\infty} \frac{n^3}{n!}$

(15) $\sum_{n=0}^{\infty} \frac{n^4}{n!} = 15e = \sum_{n=1}^{\infty} \frac{n^4}{n!}$.

Logarithmic series

An expansion for $\log_e(1+x)$ as a series of powers of x which is valid only when, $|x|<1$.

Expansion of $\log_e(1+x)$: If $|x| < 1$, then,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \text{......} \infty$$

Some important results from the logarithmic series

(1) Replacing x by $-x$ in the logarithmic series, we get

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty \text{ or, } -\log_e(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\infty$$

(2) (i) $\log_e(1+x) + \log_e(1-x) = \log_e(1-x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \ldots\ldots\ldots\infty\right), \ (-1 < x < 1)$

(ii) $\log_e(1+x) - \log_e(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\infty\right)$

$$\Rightarrow \log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\infty\right)$$

(3) The series expansion of $\log_e(1+x)$ may fail to be valid if $|x|$ is not less than 1. It can be proved that the logarithmic series is valid for $x = 1$. Putting $x = 1$ in the logarithmic series, we get

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots\infty = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots\ldots\infty$$

(4) When $x = -1$, the logarithmic series does not have a sum. This is in conformity with the fact that $\log(1-1)$ is not a finite quantity.

Difference between the exponential and logarithmic series

(1) In the exponential series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\infty$, all terms carry positive signs whereas in the logarithmic series $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$, the terms are alternatively positive and negative.

(2) In the exponential series the denominators of the terms involve factorials of natural numbers. But in the logarithmic series the terms do not contain factorials.

(3) The exponential series is valid for all the values of x. The logarithmic series is valid when $|x| < 1$.