Binomial Theorem

The rule by which any power of a binomial can be expanded is called the binomial theorem.

$$(x+a)^n = x^n + {}^nC_1\ x^{n-1}\ a + {}^nC_2\ x^{n-2}\ a^2 + \ldots\ldots\ldots..+{}^n C_r\ x^{n-r}\ a^r + \ldots.. + a^n$$

where, n is a $+ve$ integer, and ${}^nC_1, {}^nC_2 \ldots.. {}^nC_r$ are called binomial coefficients.

Some fundamental properties of binomial theorem

(1) $(x+a)^n = \sum_{r=0}^{n} {}^nC_r\ x^{n-r}\ a^r$ since r can have the values from 0 to n. The total number of terms in the expansion is $(n+1)$.

(2) The sum of the indices of x and a in each term is n.

(3) Since ${}^nC_r = {}^nC_{n-r},\ r = 0\ \ 1,2\ldots..n,\ {}^nC_0 = {}^nC_n,\ {}^nC_1 = {}^nC_{n-1},\ {}^nC_2 = {}^nC_{n-2},\ldots.$ so the beginning and the end are equal. These coefficients are known as binomial coefficients.

(4) The p^{th} term from end in the binomial expansion $(x+a)^n$ is equal to p^{th} term from the beginning in the binomial expansion $(a+x)^n$.

(5) (i) $(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2}\ a^2 + \ldots\ldots..]$ (ii) $(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3}\ a^3 + \ldots\ldots..]$

Note that $(x+a)^n + (x-a)^n$ consists of the sum of all even terms and $(x+a)^n - (x-a)^n$ consists of the sum of all odd terms.

Thus, if n is even, then $(x+a)^n + (x-a)^n$ has $\left(\frac{n}{2}+1\right)$ terms, whereas $(x+a)^n - (x-a)^n$ contains $\frac{n}{2}$ terms.

If n is odd, then $(x+a)^n + (x-a)^n$ and $(x+a)^n - (x-a)^n$ both contain $\left(\frac{n+1}{2}\right)$ terms.

General term in a binomial expansion

$$(x+a)^n = {}^nC_0\ x^n a^0 + {}^nC_1\ x^{n-1}\ a^1 + {}^nC_2\ x^{n-2} a^2 + \ldots\ldots\ldots\ \ldots + {}^nC_r\ x^{n-r}\ a^r + \ldots\ldots\ldots..+{}^nC_n\ x^0\ a^n$$

Let T_{r+1} denote the $(r+1)^{th}$ terms

$\therefore \quad T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

This is called the general term, because by giving different values to r we can determine all terms of the expansion.

In the binomial expansion of $(x-a)^n$ the general term is given by $T_{r+1} = (-1)^r\ {}^nC_r\ x^{n-r}\ a^r$

In the binomial expansion of $(1+x)^n$, $T_{r+1} = {}^nC_r x^r$

In the binomial expansion of $(1-x)^n$, $T_{r+1}=(-1)^r\ {}^nC_r x^r$

In the binomial expansion of $(x+a)^n$, the rth term from the end is $((n+1)-r+1)=(n-r+2)^{\text{th}}$ term from the beginning.

Middle terms in binomial expansion

Since the binomial expansion of $(x+a)^n$, contains $(n+1)$ terms, so

(1) If n is even. Then $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

(2) If n is odd. Then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms are two middle terms.

To find a particular term in the binomial expansion

Let in the expansion of $\left(ax^p \pm \frac{k}{x^q}\right)^n$, x^m occur in T_{r+1}, then the value of r can be obtained by using the relation $np-r(p+q)=m$ and coefficient of $x^m={}^nC_r\, a^{n-r}(\pm k)^r$. Also if in the above expansion the T_{r+1} is independent of x, then $np-r(p+q)=0$.

Some important results : In solving problems relating the coefficients in the binomial expansion, we generally use the following results

(1) Coefficient of $(r+1)^{th}$ term in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(2) Coefficient of x^r in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(3) Coefficient of x^r in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

(4) Coefficient of $(r+1)^{th}$ term in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

(5) If n is even, then ${}^nC_{n/2}$ is the greatest binomial coefficient in the expansion of $(x+a)^n$. If n is odd, then ${}^nC_{(n+1)/2}$ or ${}^nC_{(n-1)/2}$ is the greatest binomial coefficient.

Algorithm for finding the greatest term

Step 1 : Write T_{r+1} and T_r from the given expansion

Step 2 : Find $\frac{T_{r+1}}{T_r}$

Step 3 : Put $\frac{T_{r+1}}{T_r}>1$

Step 4 : Solve the inequality in step 3 for r to get inequality of the form $r<m$ or $r>m$

If m is an integer, then m^{th} and $(m+1)^{\text{th}}$ terms are equal in magnitude and these two are the greatest terms. If m in not an integer, then obtain the integral part of m, say k. In this case, $(k+1)$ th term is the greatest term.

Some important deductions

(1) Replacing n by $-n$ in the expansion for $(1+x)^n$, we get

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots\dots\dots + (-1)^r \frac{n(n+1)(n+2)\dots\dots.(n+r-1)}{r!}x^r + \dots$$

The general term in this expansion is $T_{r+1} = (-1)^r \frac{n(n+1)(n+2)\dots\dots(n+r-1)}{r!}x^r$

(2) Replacing x by $-x$ and n by $-n$ in the expansion of $(1+x)^n$, we get

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots\dots + \frac{n(n+1)(n+2)\dots.(n+r-1)}{r!}x^r + \dots$$

The general term in this expansion is $T_{r+1} = \frac{n(n+1)(n+2)\dots\dots(n+r-1)}{r!}x^r$.

(3) Replacing x by $-x$ in the expansion of $(1+x)^n$, we get

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots\dots + (-1)^r \frac{n(n-1)(n-2)\dots.(n-r+1)}{r!}x^r + \dots$$

The general term is $T_{r+1} = \frac{(-1)^r n(n-1)(n-2)\dots\dots(n-r+1)}{r!}x^r$.

(4) Putting n= 1,2,3 in the expansion of $(1+x)^{-n}$, we get

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots\dots + (-1)^r x^r + \dots\dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \dots\dots + (-1)^r (r+1)\ x^r + \dots\dots,$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots\dots + (-1)^r \frac{(r+1)(r+2)}{2} x^r + \dots\dots$$

Replacing x by $-x$ in the above expansions, we get $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots\dots + x^r + \dots\dots$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots\dots + (r+1)\ x^r + \dots\dots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + \dots\dots + \frac{(r+1)(r+2)}{2} x^r + \dots\dots.$$

Properties of binomial coefficients

$\because (1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ .....(i)

(1) **Sum of coefficients :** Putting $x = 1$ in (i) we get, $C_0 + C_1 + C_2 + \dots + C_n = 2^n$ .....(ii)

(2) **Sum of coefficients with alternate sign :** Putting $x = -1$ in (i), we get

$C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0$ .....(iii)

(3) **Sum of coefficients of even and odd terms :**

From (iii), we have $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots$ .....(iv)

i.e., $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

(4) **Sum of product of coefficients :** Replacing x by $\frac{1}{x}$ in (i), we get

$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots\dots + \frac{C_n}{x^n}$ .....(v)

Multiplying (i) by (v) we get, $\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + ...)\times\left(C_o + \frac{C_1}{x} + \frac{C_2}{x_2} + ...\right)$

Now comparing coefficient of x^r on both sides,

$$C_0C_r + C_1C_{r+1} + ... + C_{n-r}C_n = {}^{2n}C_{n+r} = \frac{2n!}{(n+r)!(n-r)!} \quad .....(vi)$$

(5) **Sum of squares of coefficients :** Putting r = 0 in (vi) we get $C_o^2 + C_1^2 + ... + C_n^2 = \frac{2n!}{n!n!}$.

(6) $C_1 + 2C_2 + 3C_2 + ...\ {}^nC_n = n2^{n-1}$ and $C_1 - 2C_2 + 3C_2 - ...... = 0$

Number of terms in the expansion of $(x + y + z)^n$

Since, $(x+y+z)^n = (x+y)^n + {}^nC_1(x+y)^{n-1}z + {}^nC_2(x+y)^{n-2}z^2 + ... + z^n$

$= (n+1)$ terms $+n$ terms $+(n-1)$ terms +.....+1 term

$\therefore$ Total number of terms in the expansion of $(x+y+z)^n = (n+1) + n + (n-1) + ... + 1$

$$= \frac{(n+1)(n+2)}{2}.$$

Note : ❑ Number of distinct terms in the $(a_1 + a_2 + ... + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.