CONICS

Contents

3. CONICS

CONIC SECTIONS

Conics or conic sections are the curves corresponding to various plane sections of a right circular cone by cutting that cone in different ways.

Each point lying on these curves satisfies a special condition, which actually leads us towards the mathematical definition of conic sections.

If a point moves in plane in such a way that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line, always remains constant, then the locus of that point is called a **Conic Section**.
A **conic section** or conic is the locus of a point, which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always a constant quantity.
The fixed point is called the focus and the fixed line is called directrix of the conic. The constant ratio is called the **eccentricity** and is denoted by e.

According to the value of eccentricity three types of conic exist i.e. for $e = 1$, $e < 1$ and $e > 1$ the corresponding conic is called **parabola**, **ellipse** and **hyperbola** respectively.

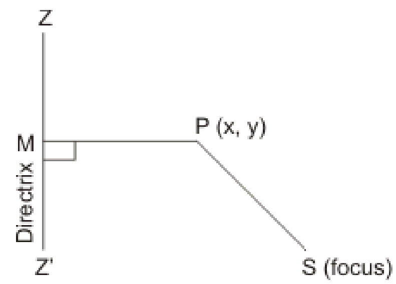
Important Terms:

- The straight line passing through the focus and perpendicular to the **directrix** is called the **axis** of the conic section.
- The points of intersection of a conic with its axis is called **vertex**.
- A chord passing through the focus is called **Focal chord** of the conic section.
- The chord passing through focus and perpendicular to axis is called **latus rectum**.
- Any chord of the **parabola** which is perpendicular to the axis is called double ordinate.
- The point which bisects every chord of the conic passing through it, is called **centre** of the conic section.

3.1 PARABOLA

3.1.1 Definition

The locus of the point, which moves such that its distance from a fixed point (i.e. focus) is always equal to its distance from a fixed straight line (i.e. directrix), is called **parabola.**



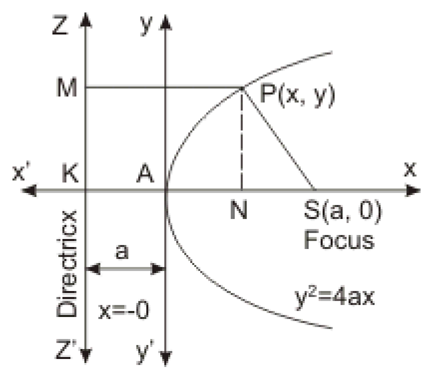
Let S be the focus, ZZ′ be the directrix and let P be any point on the parabola, then by def,

SP= PM where PM is the length of perpendicular from P on the directrix ZZ′.

3.1.2 Standard Equation of Parabola

Let S be the focus, V be the vertex, ZZ′ be the directrix and the axis of parabola be x-axis.

Draw SK perpendicular from S on the directrix and bisect SK at A. Then, AS=AK



⇒Distance of A from the focus = Distance of A from the directrix.

⇒A lies on the parabola.

Let A be the origin then coordinates of S is (a, 0).The equation of directrix ZZ′ is x=-a.

Let P(x, y) be any point on the parabola. Draw PM and PN perpendiculars on the directrix ZZ′ and x-axis. Then, PM=NK=AN+AK=x + a

Now P lies on the parabola, SP=PM (def. of Parabola)

⇒ $SP^2 = PM^2$

⇒ $(x-a)^2+(y-0)^2=(x+ a)^2$

⇒ $y^2 = 4ax$

This is the standard equation of a parabola.

There are four common forms of parabola according to their axis, with their vertex at origin

(0, 0).

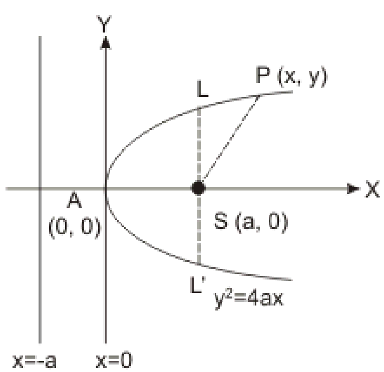
3.1.2.1 Important Terms

- The **eccentricity e** of the parabola is e = 1.
- The line y=0 (i.e., x-axis) is the **axis** of the parabola $y^2=4ax$.
- The point A(0,0) is the point of intersection of the parabola and its axis, hence, A is called the **vertex** of the parabola.
- S (a, 0) is the **focus** of the parabola.
- Corresponding to this focus, equation of **directrix** is $x = -a$.
- A chord passing through its focus is called a **focal chord**.
- Distance of any point P on the parabola from the focus is **focal distance**.

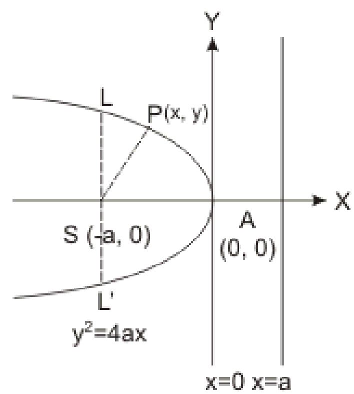
  $\therefore$Focal distance $=|SP|=|PM|$ i.e., Distance of P from directrix

  $\Rightarrow |SP|=x+a$
- A chord passing through a point P on the parabola perpendicular to the axis of the parabola is called the double ordinate through the point P.
- The double ordinate passing through the focus is called Latus-rectum. The length of latus rectum of a parabola = LL′= 2(2a) = 4a.

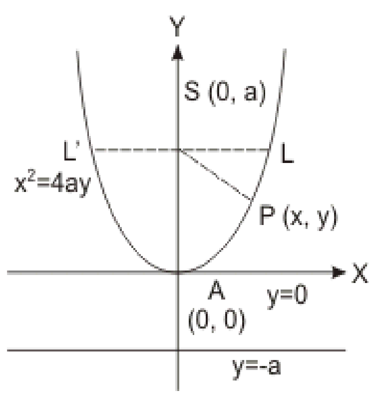
3.1.2.2 Standard Forms of Parabola



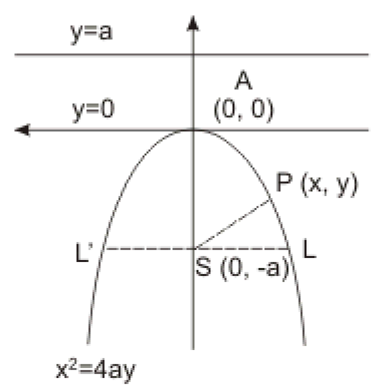
fig(i)



fig(ii)



fig(iii)



fig(iv)

Standard Equation	$y^2 = 4ax (a > 0)$	$y^2 = -4ax (a > 0)$	$x^2 = 4ay (a > 0)$	$x^2 = -4ay (a > 0)$
Shape of the parabola	See fig(i)	See fig(ii)	See fig(iii)	See fig(iv)
Vertex	A (0,0)	A (0,0)	A (0,0)	A (0,0)
Focus	S (a,0)	S (-a,0)	S (0,a)	S (0,-a)
Equation of directrix	x = -a	x=a	y=-a	y=a
Equation of axis	y=0	y=0	x=0	x=0
Length of latus rectum	4a	4a	4a	4a
Extremities of latus rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Equation of latus rectum	x=a	x=–a	y=a	y=–a
Focal distance of a point P(x, y)	x + a	a– x	a+ y	a–y
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$

If the vertex is not origin and the axis and directrix are parallel to the coordinate axes, then the equation of parabola with vertex at (h, k) can be obtained by using translation of axes as follows:

Form	$(y-k)^2 = 4a(x-h)$	$(y-k)^2 = -4a(x-h)$	$(x-h)^2 = 4a(y-k)$	$(x-h)^2 = -4a(y-k)$
Vertex	(h, k)	(h, k)	(h, k)	(h, k)
Focus	(h+ a, k)	(h–a, k)	(h, k + a)	(h, k–a)
Equation of directrix	x = h–a	x=h+ a	y=k–a	y=k+a
Equation of axis	y=k	y=k	x=h	x=h

Illustration: Find the vertex, axis, directrix, and the length of the latus rectum of the parabola $2y^2 + 3y - 4x - 3 = 0$.

Solution: The given equation can be re-written as $(y–3/4)^2= 2\ (x+33/32)$

which is of the form $y^2=4ax$.

Hence the vertex is $(-33/32, -3/4)$.

The axis is $y + 3/4 = 0 \Rightarrow y = -3/4$.

The directrix is $x + a = 0$.

$\Rightarrow x + 33/32+1/2 = 0 \Rightarrow x = -\ 49/32$.

Length of the latus rectum $= 4a = 2$.

3.1.2.3 Position of a Point Relative to a Parabola

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =$ or < 0, respectively.

Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

3.1.2.4 Intersection of a line and Parabola

- The line $y= mx + c$ intersects the parabola $y^2=4ax$ in two distinct points if $c < \frac{a}{m}$.
- The line $y= mx + c$ intersects the parabola $y^2=4ax$ in one point(i.e., touches) if $c = \frac{a}{m}$
- The line $y= mx + c$ does not intersects the parabola $y^2=4ax$ if $c \geq \frac{a}{m}$

Note:

When m is very small, one of the roots of equation (i) is very large; when m is equal to zero, this root is infinitely large. Hence every straight line parallel to the axis of the parabola meets the curve in one point at a finite distance and in another point at an infinite distance from the vertex. It means that a line parallel to the axis of the parabola meets the parabola only in one point.

3.1.3 Parametric Equation of a Parabola

For Parabola $y^2=4ax$, for any real t, $x=at^2$, $y=2at$ satisfy the equation of parabola. So the point $(at^2, 2at)$ lies on the parabola.

Here $x= at^2$, $y=2at$ is known as parametric form of the parabola. This point is also called the point t denoted by P (t).

Note: The Points $\left(\frac{a}{t^2}, \frac{2a}{t}\right)$ as well as $\left(at^2, -2at\right)$ also lie on the parabola.

Illustration: Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $a^2\ |(t_1 - t_2)\ (t_2 - t_3)\ (t_3 - t_1)|$ where t_1, t_2 and t_3 are the vertices.

Solution: The three points on the parabola are $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$.

Hence, area, $\Delta = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix} = a^2\begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 - t_1^2 & t_2 - t_1 & 0 \\ t_3^2 - t_2^2 & t_3 - t_2 & 0 \end{vmatrix}$

$= |a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$

Illustration: Find the equation of the parabola whose focus is (3, –4) and directrix is $x - t + 5 = 0$.

Solution: Let P(x, y) be any point on the parabola.

Then

$$\sqrt{(x-3)^2 + (y+4)^2} = \frac{|x-y+5|}{\sqrt{1+1}}$$

$\Rightarrow (x - 3)^2 + (y + 4)^2 = (x-y+5)^2/2$

$\Rightarrow x^2 + y^2 + 2xy - 22x + 26y + 25 = 0.$

$\Rightarrow (x + y)^2 = 22x - 26y - 25.$

Illustration: Find the equation of the parabola having focus (–6, 6) and vertex (–2, 2).

Solution: Let S(–6, –6) be the focus and A(–2, 2) the vertex of the parabola.

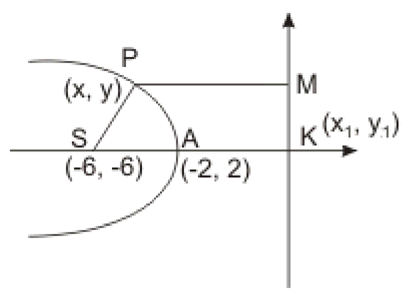
On SA take a point K (x_1, y_1) such that SA = AK.

Draw KM perpendicular on SK. Then KM is the directrix of the parabola**.**

Since A bisects SK, $((-6+x_1)/2, (-6+y_1)/2) = (-2, 2)$

$\Rightarrow -6 + x_1 = -4$, and $-6 + y_1 = 4$

Or $(x_1, y_1) = (2, 10)$.



Hence the equation of the directrix KM is $y - 10 = m(x + 2)$. … (1)

Also gradient of

SK = (10-(-6))/(2-(-6))=16/8 = 2;

$\Rightarrow$ Slope of directrix, m = (-1)/2

So that equation (1) becomes $y - 10 = -1/2\ (x - 2)$

or $x + 2y - 22 = 0$ is the directrix.

let PM be a perpendicular on the directrix KM from any point P(x, y) on the parabola.

From SP = PM, the equation of the parabola is

$\sqrt{(\{(x+6)+(y+6)^2\})}=(x+2y-22)/((1^2+2^2))$

or $5 = (x^2 + y^2 + 12x + 12y + 72) = (x + 2y - 22)^2$

or $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$.

or $(2x - y)^2 + 104x + 148y - 124 = 0$.

Illustration: If the point (2, 3) is the focus and $x = 2y + 6$ is the directrix of a parabola, find

(i) the equation of the axis,

(ii) the co-ordinates of the vertex,

(iii) length of the latus rectum,

(iv) equation of the latus rectum.

Solution: (i) We know that the axis of a parabola is the line through the focus and perpendicular to the directrix.

The equation of any line passing through the focus (2, 3) is

$y - 3 = m(x - 2) \Rightarrow mx - y = 3 - 2m$

If the line be perpendicular to the directrix $x - 2y = 6$, we have,

$m(1/2) = -1 \Rightarrow m = -2$.

Hence the equation of the axis is $y - 3 = -2(x - 2) \Rightarrow 2x + y = 7$.

(ii) The co-ordinates of the point of intersection (say) A of the directrix $x - 2y = 6$ and the axis $2x + y = 7$ are obtained by solving the two equations; thus they are $(4, -1)$.

Since the vertex is the middle point of A $(4, -1)$ and the focus S (2, 3); the co-ordinates of the vertex are $((4+2)/2, (3-1)/2)$, i.e. (3, 1).

(iii) Since $OS = \sqrt{((3-2)^2+(1-3)^2)}=\sqrt{5}$,

The length of the latus rectum = $4OS = 4\sqrt{5}$.

(iv) Since the latus rectum is the line through the focus parallel to the directrix, its equation is $x - 2y + c = 0$, where c is given by $2 - 2(3) + c = 0$, i.e. $c = 4$.

Illustration: Prove that the circle with any focal chord of the parabola $y^2 = 4ax$ as its diameter touches its directrix.

Solution: Let AB be a focal chord. If A is $(at^2, 2at)$ then B is $(a/t^2, -2a/t)$

Equation of the circle with AB as diameter is

$(x - at^2)(x-a/t^2) + (y - 2at)(y+2a/t) = 0$.

For $x = -a$, this gives $(a^2 (1+t^2)^2)/t^2 + y^2 - 2ay (t-1/t) - 4a^2 = 0$.

$\Rightarrow a^2 (t-1/t)^2 + y^2 - 2ay (t - 1/t) = 0$

$\Rightarrow [y - a (t - 1/t)]^2 = 0$, which has equal roots.

Hence $x + a = 0$ is a tangent to the circle with diameter AB.

Illustration: Find the locus of the centre of the circle described on any focal chord of a parabola as diameter.

Solution: Let the equation of the parabola be $y^2 = 4ax$.

Let t_1, t_2 be the extremities of the focal chord. Then $t_1 t_2 = -1$.

The equation of the circle on t_1, t_2 as diameter is

$(x - at_1^2) (x - at_2^2) + (y - 2at_1) (y - 2at_2) = 0$

or $x^2 + y^2 - ax (t_1^2 + t_2^2) - 2ay (t_1 + t_2) + a^2 t_1^2 t_2^2 + 4a^2 t_1t_2 = 0$

$\Rightarrow x^2 + y^2 - ax (t_1^2 + t_2^2) - 2ay (t_1 + t_2) - 3a^2 = 0$. ($\because t_1t_2 = -1$)

If (α, β) be the centre of the circle, then

$\alpha = a/2\, (t_1^2+t_2^2)$

$\beta = a (t_1 + t_2) \Rightarrow (t_1 + t_2)^2 = \dfrac{\beta^2}{a^2}$

$\Rightarrow t_1^2 + t_2^2 + 2t_1t_2 = \dfrac{\beta^2}{a^2}$

$\Rightarrow \dfrac{2\alpha}{a} - 2 = \beta^2/a^2$

$\Rightarrow 2a\alpha - 2a^2 = \beta^2$

$\Rightarrow \beta^2 = 2a (\alpha - a)$.

Hence locus of (α, β) is $y^2 = 2a(x - a)$.

3.1.4 Chords of Parabola

3.1.4.1 Equation of Chord

- Equation of chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola $y^2=4ax$ is $y(y_1+y_2) = 4ax + y_1y_2$
- Equation of chord joining any two points i.e., $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ whose parameters are t_1 and t_2 is $y(t_1+t_2) = 2\left(x + at_1t_2\right)$

If the equation of chord is $y = mx + c$ then

- Coordinates of the midpoint of the chord are $\left(\dfrac{2a - mc}{m^2}, \dfrac{2a}{m}\right)$
- The length of the chord is given by $\dfrac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a - mc)}$

Equation of a Chord Bisected at a Given Point:

The equation of a chord of Parabola $y^2=4ax$ bisected at (x_1, y_1) is given by $T = S_1$ where $T \equiv yy_1 - 2a(x+x_1)$ and $S_1 \equiv y_1^2 - 4ax_1$.

Note: Equation of chord bisected at a given point also means equation of chord with its midpoint.

Illustration: Find the length of the chord intercepted by the parabola $y^2 = 4ax$ from the line $y = mx + c$. Also find its mid-point.

Solution: Simply by applying the formula of length of the joining (x_1, y_1) and (x_2, y_2) we get,

Length of the chord = $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

= $\sqrt{(x_1-x_2)^2+m(x_1-x_2)^2}$

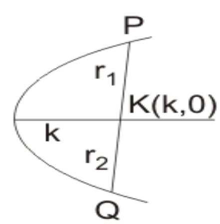
= $|x_1 - x_2|\sqrt{(1+m^2)} = 4\sqrt{(a(a-mc))}\sqrt{(1+m^2)}$

$[\because x_1+x_2=(-2(m-2a))/m^2$ and $x_1 x_2=c^2/m^2]$

The midpoint of the chord is $((2a-mc)/m^2, 2a/m)$

Illustration: Prove that for a suitable point K on the axis of the parabola the chord PQ, passing through K, can be drawn such that $\frac{1}{PK^2}+\frac{1}{QK^2}$ is the same for all positions of the chord.

Solution: The chord PQ passes through K (k, 0) where k is to be suitably taken.
Any point on PQ is $(k + r\cos\theta, r\sin\theta)$ where $\tan\theta$ is the slope of PQ.
The equation of the parabola is $y^2 = 4ax$.



The point $(k + r\cos\theta, r\sin\theta)$ is on $y^2 = 4ax$ if

$r^2\sin^2\theta = 4a(k + r\cos\theta)$

or $r^2\sin^2\theta - 4ar\cos\theta - 4ak = 0$ ...(1)

If $KP = r_1$, $KQ = r_2$ then r_1, r_2 are roots of (1).

$\therefore \quad r_1 + r_2 = \frac{4a\cos\theta}{\sin^2\theta}, \quad r_1r_2 = \frac{-4ak}{\sin^2\theta}$

Now, $\frac{1}{PK^2}+\frac{1}{QK^2} = \frac{1}{r_1^2}+\frac{1}{r_2^2}$

$= \frac{r_1^2+r_2^2}{r_1^2r_2^2} = \frac{(r_1+r_2)^2-2r_1r_2}{(r_1r_2)^2}$

$$= \frac{\cos^2\theta}{k^2} - 2.\frac{\sin^2\theta}{-4ak} = \frac{\cos^2\theta}{k^2} + \frac{\sin^2\theta}{2ak}$$

$$= \frac{\cos^2\theta}{4a^2} + \frac{\sin^2\theta}{4a^2} \text{ if } k = 2a$$

$$= \frac{1}{4a^2}, \text{independent of } \theta.$$

$\therefore \quad \frac{1}{PK^2} + \frac{1}{QK^2}$ is the same for all positions of the chord if $K = (2a, 0)$.

3.1.4.2 Condition for the chord to be focal chord

The equation of chord is $y(y_1 + y_2) - 4ax - y_1y_2 = 0$

1. Condition for the chord joining 'y_1' and 'y_2' to pass through the focus (a, 0) is obtained by putting x=a and y=0 in the above equation

   $\therefore -4a^2 - y_1y_2 = 0 \quad \text{or} \quad y_1y_2 = -4a^2$

2. Condition for the chord joining 't_1' and 't_2' to be a focal chord is t_1t_2=-1 or $t_2 = \frac{1}{t_1}$.

Note: Hence if one extremity of a focal chord is ($at_1^2, 2at_1$), then the other extremity ($at_2^2, 2at_2$) becomes $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$.

3.1.4.3 Diameter of Parabola

The locus of the middle point of a system of parallel chords of a parabola is called its diameter.

Equation of diameter bisecting chords of slope m of the parabola y^2=4ax is $y = \frac{2a}{m}$.

Note: y = 2a/m is a straight line parallel to the axis of the parabola.

3.1.5 Tangent and Normal

The condition for the line y= mx+c to be a tangent to the parabola y^2=4ax is $c = \frac{a}{m}$ and the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Note:

- The angle between the tangents drawn to the two parabolas at the point of their intersection is defined as the angle of intersection of two parabolas.
- The **point of intersection of tangents** drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to the parabola y^2=4ax is R≡ (at_1t_2, a (t_1 + t_2)).

3.1.5.1 Equation of Tangent in Different forms

Point Form:

The equation of tangent to the parabola $y^2=4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$ i.e., T = 0 where

T = $yy_1 - 2a(x + x_1)$.

Parametric Form:

The equation of tangent to the parabola $y^2=4ax$ at $(at^2, 2at)$ is $ty = x + at^2$.

Slope Form:

The equation of tangent to the parabola $y^2=4ax$ in terms of slope m is $y = mx + \frac{a}{m}$ and

coordinate of the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Illustration: On the parabola $yy^2 = 4ax$, three points E, F, G are taken so that their ordinates are in G.P. Prove that the tangents at E and G intersect on the ordinate of F.

Solution: Let the points E, F, G be $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$ respectively.

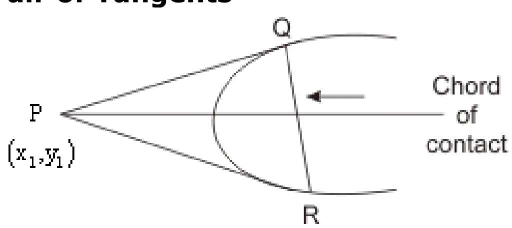
Since the ordinates of these points are in G.P., $t_2^2 = t_1t_3$.

Tangents at E and G are $t_1y = x + at_1^2$ and $t_3y = x + at_3^2$.

Eliminating y from these equation, we get $x = at_1t_3 = at_2^2$.

Hence the point lies on the ordinates of F.

3.1.5.2 Equation of Pair of Tangents



The Equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2=4ax$ is $\mathbf{SS_1=T^2}$ Where, $S \equiv y^2 - 4ax$, $S_1 \equiv y_1^2 - 4ax_1$ and $T \equiv yy_1 - 2a(x + x_1)$.

3.1.5.3 Chord of Contact

If the tangents from the external point (x_1, y_1) touch the parabola at Q and R, then QR is the chord of contact of the tangents.

The **equation of chord of contact** of tangents drawn from a point $P(x_1, y_1)$ to the Parabola $y^2=4ax$ is T=0 where $T \equiv yy_1 - 2a(x + x_1)$.

Illustration: Find the equation of the chord of the parabola $y^2 = 12x$ which is bisected at the point (5, –7).

Solution: Here $(x_1, y_1) = (5, -7)$, and $y^2 = 12x = 4ax \Rightarrow a = 3$.

The equation of the chord is $S_1 = T$

or $y_1^2 - 4ax_1 = yy_1 - 2a(x + x_1)$

$(-7)^2 - 12.5 = y(-7) - 6(x + 5)$

$6x + 7y + 19 = 0.$

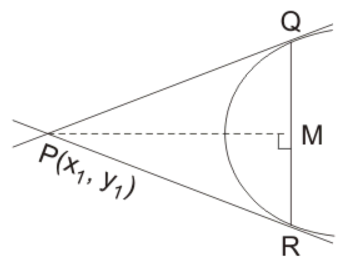
Illustration: Prove that the area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\dfrac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a}$.

Solution: Equation of QR (chord of contact) is

$yy_1 = 2a(x + x_1)$

or $yy_1 - 2a(x + x_1) = 0$

$\because$ PM = length of perpendicular from $P(x_1, y_1)$ on QR



$$= \frac{|y_1y_1 - 2a(x_1 + x_1)|}{\sqrt{(y_1^2 + 4a^2)}}$$

$$= \frac{(y_1^2 - 4ax_1)}{\sqrt{(y_1^2 + 4a^2)}}$$ {Since $P(x_1, y_1)$ lies outside the parabola $\Rightarrow y^2{}_1 - 4ax_1 > 0$}

Now area of $\Delta PQR = \dfrac{1}{2}\, QR.\, PM$

$$= \frac{1}{2}.\frac{1}{|a|}.\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}.\frac{(y_1^2 - 4ax_1)}{\sqrt{(y_1^2 + 4a^2)}}$$

$$= \frac{(y_1^2 - 4ax_1)^{3/2}}{2a}, \text{ if } a > 0.$$

3.1.5.4 Equation of Normal in Different forms

The normal to a curve is a line perpendicular to the tangent to curve through the point of contact.

Point Form:

The equation of the normal to the parabola $y^2=4ax$ at the point (x_1, y_1) is $y - y_1 = -\dfrac{y_1}{2a}\left(x - x_1\right)$.

Parametric Form:

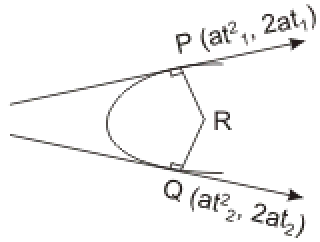
The equation of the normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is $y + tx = 2at + at^3$.

Slope Form:

The equation of the normal to the parabola $y^2=4ax$ in terms of slope m is $y = mx - 2am - am^3$.

Important Points

- The line $y = mx + c$ is normal to the parabola $y^2=4ax$ if $c=-2am-am^3$.
- The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ a $Q(at_2^2, 2at_2)$ to the parabola $y^2=4ax$ is



R≡ $(2a + a(t_1^2+t_2^2 + t_1 t_2), - at_1t_2(t_1 + t_2))$.

- If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$ then $t_2 = -t_1 - \frac{2}{t_1}$

  Note: PQ is normal to the parabola at P and not at Q.

- If the normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet on the parabola $y^2=4ax$ then $t_1t_2=2$.

Illustration: P, Q are the points with parameters t_1, t_2 on the parabola $y^2 = 4ax$. If the normals at P, Q meet on the parabola at $R(t_3)$ then show that $t_1t_2 = 2$ and

$t_3 = -t_1 - t_2$.

Solution: Let the normals at P, Q meet at $R(x_1, y_1)$ where $x_1 = at_3^2$, $y_1 = 2at_3$.
The normal to the parabola at 't' is $y + tx = 2at + at^3$
If it passes through $R(x_1, y_1)$, then $y_1 + tx_1 = 2at + at^3$
or $at^3 + (2a - x_1)t - y_1 = 0$
If t_1, t_2, t_3 are the roots of equation (i), then

$$t_1t_2t_3 = -\frac{\text{constant term}}{\text{coefficient of } t^3} = -\left(\frac{-y_1}{a}\right) = \frac{y_1}{a} = \frac{2at_3}{a} = 2t_3$$

$\therefore$ $t_1t_2 = 2$.
Also, $t_1 + t_2 + t_3 = 0 \Rightarrow t_3 = -t_1 - t_2$.

3.1.5.5 Co-normal Points

Any three points on the normals of a parabola pass through a common point are called co-normal points.

If three normals are drawn through a point (h, k) then their slopes are the roots of the cubic $k = mh - 2am - am^3$

- The sum of the slopes of the normals at co-normal points is zero, i.e. $m_1 + m_2 + m_3 = 0$
- The sum of the ordinates of the co-normal points is zero

  $(i.e., -2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0)$

- The centroid of the triangle formed by the co-normal point's lies on the axis of the parabola [the vertices of the triangle formed by the co-normal points are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$. Thus, y-coordinate of the centroid becomes $\frac{-2a(m_1+m_2+m_3)}{3}=\frac{-2a}{3}\times 0=0$. Hence, the centroid lies on the x-axis of the parabola]

- If three normals drawn to any parabola $y^2=4ax$ from a given point (h, k) be real, then h>2a.

Illustration: Find the locus of the point of intersection of two normals to a parabola which are at right angles to one another.

Solution: The equation of the normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$.

It passes through the point (h, k) if $k = mh - 2am - am^3$

$\Rightarrow am^3 + m(2a - h) + k = 0$. … (1)

Let the roots of the above equation be m_1, m_2 and m_3. Let the perpendicular normals correspond to the values of m_1 and m_2 so that $m_1 m_2 = -1$.

From equation (1), $m_1 m_2 m_3 = -k/a$. Since $m_1 m_2 = -1$, $m_3 = k/a$.

Since m_3 is a root of (1), we have

$a(k/a)^3 + k/a(2a - h) + k = 0. \Rightarrow k^2 + a(2a - h) + a^2 = 0$

$\Rightarrow k^2 = a(h - 3a)$.

Hence the locus of (h, k) is $y^2 = a(x - 3a)$.

Illustration: Prove that the normal chord to a parabola at the point whose ordinate is equal to the abscissa subtends a right angle at the focus.

Solution: If the normal to the parabola $y^2 = 4ax$ at $P(at_1t_2, 2at_1)$ meets it again at the point t_2, then we have $t_2 = -t_1 - 2/t_1$.

If the abscissa and the ordinates of P be equal, then $at_1^2 = 2at_1$

$\Rightarrow t_1 = 2$ (rejecting $t_1 = 0$).

$\therefore$ The co-ordinates of P and Q are (4a, 4a) and (9a, –6a) respectively.

The focus is the point S (a, 0).

Slope of PS = ¾ and slope of QS = –¾.

$\Rightarrow \angle PSQ$ = right angle.

Hence it subtends right angle at the focus.

Illustration: Find the locus of the middle points of normal chords of the parabola$y^2 = 4ax$.

Solution: Equation of the normal chord at any point $(at_2, 2at)$ of the parabola is

$y + tx = 2at + at^3$. … (1)

Equation of the chord with mid point (x_1, y_1) is T = S1

or $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$ or $yy_1 - 2ax = y_1^2 - 2ax_1$. … (2)

Since equations (1) and (2) are identical,

$1/y_1 = t/(-2a) = (2at+at^3)/t = 2a + ((-2a)/y_1)^2$

or $-(y_1^2)/2a + x_1 = 2a + 4a^3/(y_1^2)$ or $x_1 - 2a = (y_1^2)/2a + 4a^3/(y_1^2)$

Hence the locus of the middle point (x_1, y_1) is

$x - 2a = y^2/2a + 4a^3/y^2$.

Illustration: Find the equations of the normals to the parabola $y^2 = 4ax$ at the extremities of its latus rectum. If the normals meet the parabola, again at P and Q, prove that PQ = 12a.

Solution: The ends of the latus rectum are (a, 2a) and (a, –2a). The equations of the normals to the parabola at these points are (put t = 1 and –1)

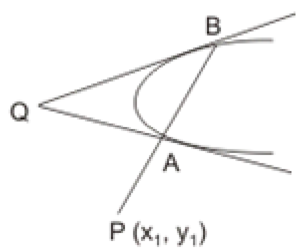
$y + x = 3a$ and $y - x = 3a$.

These lines meet the parabola again at P(9a, 6a) and Q(9a, 6a) respectively.

$\Rightarrow$ PQ = 6a + 6a = 12a.

3.1.6 Pole and Polar

The locus of the points of intersection of tangents drawn at the point extremities of the chords passing through a fixed point is called the polar of that fixed point and the fixed point is called the pole.



Let P be (x_1, y_1) and parabola be $y^2=4ax$. Tangents to the Parabola at A and B meet at Q (h, k). AB is the chord of contact with respect to Q (h, k).

$\therefore$ Equation of AB is $ky = 2a(x + h)$

But $P(x_1, y_1)$ lies on it, hence $ky_1 = 2a(x_1 + h)$

Hence locus of Q(h, k) i.e., polar of the point P is $yy_1 = 2a(x + x_1)$

Note:

- Any tangent is the polar of its point of contact.
- The polar of the focus is directrix.
- If the polar $P(x_1, y_1)$ passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to be conjugate points.

3.1.7 Solved Examples

Example: Find the equation of the parabola whose focus is (3, −4) and directrix is the line parallel to 6x − 7y + 9 = 0 and directrix passes through point (3/2,2).

Solution: Let (x, y) be any point on the parabola.

The line parallel to 6x − 7y + 9 = 0 is 6x − 7y + 6 = 0 …… (1)

Since directrix passes through (3/2, 2), this point will satisfy equation (1) hence

6 (3/2) − 7 (2) + k = 0

⇒ k = − 9 + 14 = 5.

Equation of directrix is 6x − 7y + 5 = 0

Then by definition, the distance between (x, y) and the focus (3, − 4) must be equal to the length of perpendicular from(x, y) on directrix.

$$\sqrt{(x-3)^2+(y+4)^2} = \left|\frac{(6x-7y+5)}{\sqrt{6^2+7^2}}\right|$$

⇒ 85 $\{(x-3)^2 + (y+4)^2\} = (6x - 7y + 5)^2$

⇒ $49x^2 + 36y^2 + 84xy - 570x + 750y + 2100 = 0$.

Example: Find the angle of intersection of the parabola $y^2 = 8x$ and $x^2 = 27y$.

Solution: The given parabolas are

$y^2 = 8x$ …… (1)

and $x^2 = 27y$ …… (2)

Solving (1) and (2) we get

$(x^2/27)^2 = 8x$

⇒ $x^4 = 5832x$

⇒ $x^4 - 5832x = 0$

⇒ $x(x^3 - 5832) = 0$

⇒ x = 0, x = 18

Substituting these values of x in (2) we get y = 0, 12

The point of intersection are (0, 0), (18, 12)

At the point (0, 0)

$y^2 = 8x \Rightarrow dy/dx=4/y \Rightarrow dy/dx|_{(0,0)} = \infty$

$x^2 = 27y \Rightarrow dy/dx=2x/27 \Rightarrow dy/dx|_{(0,0)} = 0$

⇒ the two curves intersect at the point (0, 0) at right angle.

At the point (18, 12)

$y^2 = 8x \Rightarrow dy/dx|_{(18,12)} = 4/12=1/3 = m_1$ (say)

$x^2 = 27y \Rightarrow dy/dx=2x/27 \Rightarrow dy/dx|_{(18,12)} = (2\times18)/27=4/3 = m_2$ (say)

Let θ be the angle at which the two curves intersect at the point (18, 12)

Then $\tan \theta_{acute} = |(m_2-m_1)/(1+m_1 m_2)|=3/13$

$\theta_{acute} = \tan^{-1}(3/13)$

Example: Prove that $(x + a)^2 = (y^2 - 4ax)$, is the locus of the point of intersection of the tangents to the parabola $y^2 /4\pi= 4ax$, which includes an angle.

Solution: Let two tangent to the parabola $y^2 = 4ax$ …… (1)

be $yt_1 = x + at_1^2$ …… (2)

and $yt_2 = x + at_2^2$ …… (3)

Let the point of intersection of the tangent be (x_1, y_1).

then solving equation (1)and (2) we get,

$x_1 = at_1 t_2$

$y_1 = a(t_1 + t_2)$

Also the slope of these tangents are $1/t_1$ and $1/t_2$

∴ If α be the angle between these two tangents then

$$\tan\alpha = \pm\left|\frac{(m_1 - m_2)}{(1+m_1m_2)}\right| = \pm\left|\frac{\left(\frac{1}{t_1} - \frac{1}{t_2}\right)}{\left(1+\frac{1}{t_1}\times\frac{1}{t_2}\right)}\right|$$

$\Rightarrow \tan \alpha = \pm ((t_2-t_1)/(1+t_2 t_1))$

we are given $\alpha = \pi/4$

$\therefore \tan \pi/4 = 1 = \pm ((t_2-t_1)/(1+t_2 t_1))$

$\Rightarrow (1 + t_1t_2)^2 = (t_2 - t_1)^2$

$\Rightarrow \{1+(x_1/a)\}^2 = (t_1+t_2)^2 - 4t_1t_2 = (y_1/a)^2-4x_1/a$

$\Rightarrow (x_1 + a)^2 = y_1^2 - 4ax_1$

⇒ Required locus of (x_1, y_1) is $(x + a)^2 = (y^2 - 4ax)$

Example: Prove that normal at one end of latus rectum of a parabola is parallel to the tangent at the other end.

Solution: Let the parabola be $y^2 = 4ax$ …… (1)

The end points of latus rectum are (a, 2a) & (a, −2a) and

The equation of the normal to (1) at (a, 2a) is $(y - 2a) = -(x - a)$

$\Rightarrow y = -x + 3a$ …… (2)

The equation of the tangent to (1) at (a, −2a) is $y + 2a = -(x - a)$

$\Rightarrow y = -x - a$

and from (2) and (3), we find that the slope of normal to one end of the latus rectum is equal to the slope of tangent at other end of tangent to the other end. Hence the required fact is proved.

Example: Prove that the locus of the point of intersection of two mutually perpendicular tangents one to each of the parabola $y^2 = 4a(x + a)$ and $y^2 = 4b(x + b)$, is a line parallel to y-axis.

Solution: The given parabolas are

$y^2 = 4a(x + a)$ …… (1)

and $y^2 = 4b(x + b)$ …… (2)

Any tangent to (1) is $y = m(x + a) + a/m$ …… (3)

Similarly any tangent to (2) is $y = m_1(x + b) + b/m_1$ …… (4)

$mm_1 = -1$

or $m_1 = -1/m$

$\therefore$ (4) becomes $y = -\{(x + b)/m\} - bm$ …… (5)

The required locus is obtained by eliminating m between (3) and (5).

For this subtracting (5) from (3), we get

$0 = x(m+1/m) + a(m+1/m) + b(m+1/m)$

$\Rightarrow x + a + b = 0$

This is the required locus which is parallel to y axis.

Example: Find the locus of the point P if the perpendicular from that point P upon its polar with respect to parabola, is of constant length.

Solution: Polar of P (x_1, y_1) with respect to the parabola $y^2 = 4ax$ is

$y_1y = 2a(x_1 + x)$

or $y_1y = 2ax - 2ax_1 = 0$ …… (1)

We are given that the distance of P (x_1, y_1) from line (1) is constant, say

$\Rightarrow |y_1^2 - 2ax_1 - 2ax_1| / \sqrt{(y_1^2 + (2a)^2)} = \lambda$ (constant)

$\Rightarrow$ Locus of given point P is $(y^2 - 4ax)^2 = \lambda^2 (y^2 + 4a^2)$

Example: Find the equation of the parabola whose focus is (1, 1) and tangent at the vertex is x+y=1.

Solution: Let S be the focus and A be the vertex of the parabola. Let K be the point of intersection of the axis and directrix.

Since axis is a line passing through S (1, 1) and perpendicular to x + y=1.

So, let the equation of the axis be $x - y + \lambda = 0$

This will pass through (1, 1), if

$1 - 1 + \lambda = 0 \quad \Rightarrow \lambda = 0$

So the equation of the axis is

x – y = 0 ...(i)

The vertex A is the point of intersection of x-y=0 and x + y= 1. Solving these two equations, we get x=1/2, y=1/2.

So, the coordinates of the vertex A are (1/2, 1/2).

Let (x_1, y_1) be the coordinates of k. Then, $\frac{x_1+1}{2} = \frac{1}{2}$, $\frac{y_1+1}{2} = \frac{1}{2}$

$\Rightarrow \; x_1 = 0$, $y_1 = 0$

So, the coordinates of k are (0,0).

Since directrix is a line passing through k (0, 0) and parallel to x + y=1.

Therefore, equation of the directrix is

$y - 0 = -1(x - 0) \quad \Rightarrow \quad x + y = 0.$ ...(ii)

Let P(x, y) be any point on the parabola. Then,

Distance of *P* from the focus S= [Distance of *P* from the directrix x + y=0]

$$\Rightarrow \quad \sqrt{(x-1)^2 + (y-1)^2} = \left(\frac{x+y}{\sqrt{1^2+1^2}} \right)$$

$$\Rightarrow \quad 2x^2 + 2y^2 - 4x - 4y + 4 = x^2 + y^2 + 2xy$$

$$\Rightarrow \quad x^2 + y^2 - 2xy - 4x - 4y + 4 = 0 .$$

This is the equation of the required parabola.

Example: Find the locus of the point, from which the three normals to the parabola y^2 = 4ax cut the axis at points whose distance from the vertex are in A.P.

Solution: Any normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ … (1)

If (1) passes through (x_1, y_1) then $y_1 = mx_1 - 2am - am^3$

$\Rightarrow am^3 + m(2a - x_1) + y_1 = 0$ … (2)

$m_1 + m_2 + m_3 = 0$ … (3)

$m_1m_2 + m_2m_3 + m_3m_1 = ((2a-x_1))/a$ … (4)

$m_1 m_2 m_3 = (-y_1)/a$ … (5)

Also, the normal with slope m_1, i.e. $y = m_1x - 2am_1 - am_1^3$ cuts axis of the parabola at the point $A(2a + am_1^2, 0)$

Similarly, normals with slope m_2 and m_3 cut the axis at $B(2a + am_2^2, 0)$ and $C(2a + am_3^2, 0)$.

OA, OB and OC are in A.P. (given) where O is origin or vertex of the parabola.

$\Rightarrow$ 2OB = OA + OC … (6)

$\Rightarrow 2m_2^2 = m_1^2 + m_3^2$

$\Rightarrow (m_1 + m_2 + m_3)^2 - 2(m_1 m_2 + m_2 m_3 + m_3 m_1)$

$= 0 - 2((2a-x_1)/a)$

$\Rightarrow m_2^2 = 2/3((x_1-2a)/a)$ … (7)

From equation (3), we have

$m_2^2 = (m_1 + m_3)^2$

$= m_1^2 + m_3^2 + 2 m_1 m_3$

$= 2m_2^2 + 2(-y_1/am_2)$ (using 6 and 5)

$\Rightarrow m_3^2 = -2y_1/a$ … (8)

Cubing equation (7) and squaring equation (8) we get

$27 ay_1^2 = 2(x_1 - 2a)^3$

$\therefore$ Locus of point (x_1, y_1) is

$27 ay^2 = 2(x - 2a)^3$

Example: Show that the locus of the mid point of all focal chords of a parabola is also a parabola.

Solution: Equation of chord is given by $y(t_1 + t_2) = 2(x + at_1t_2)$ … (1)

It passes through the focus (a, 0) … (2)

$\Rightarrow 0 = 2(a + at_1 + t_2^2) \Rightarrow t_1t_2 = -1$

$\Rightarrow 2h = a(t_1^2 + t_2^2)$ and $k = a(t_1 + t_2)$ … (3)

$\Rightarrow 2h = a((t_1 + t_2)^2 - 2t_1t_2)$

$= a((k/a)^2+2)$ using (2) and (3)

$\Rightarrow k^2 = 2a(h - a)$

$\therefore$ Locus of M (h, k) is $y^2 = 2a(x - a)$

$\Rightarrow$ Locus of the mid point of all focal chords of a parabola is also a parabola.

Example: A tangent to the parabola $y^2 + 12x = 0$ cuts the parabola $y^2 = 4ax$ at P and Q. Find the locus of middle points of PQ.

Solution: Any tangent to the parabola $y^2 = -4bx$ is $y = mx - b/m$

$y = mx - 3/m$

Let (x_1, y_1) be the mid point of PQ, where P and Q are point of intersection of line (1) and $y^2 = 4ax$

Equation of chord PQ is

$S_1 = T$

$y_1^2 - 4ax_1 = y_1y - 2a(x + x_1)$

$y_1y - 2ax - y_1^2 + 2\ ax_1 = 0$ …… (2)

Equation (1) can be written as

$my - m^2 x + 3 = 0$ …… (3)

Equation (2) and (3) represent the same line

$\Rightarrow m/y_1 = m^2/2a = 3/(2ax_1 - y_1^2)$ …… (4)

$\Rightarrow m = 2a/y_1$ (from 4)

Again, from (4), we get

$m/y_1 = 3/(2ax_1 - y_1^2)$

$\Rightarrow 2a/y_1 = 3/(2ax_1 - y_1^2)$

$\Rightarrow$ Locus of (x_1, y_1) is

$4a^2x = y^2 (3 + 2a)$

Example: The normal at any point P of the $y^2 = 4ax$ meets the axis in G and to the tangent at the vertex at H. If A be the vertex and the rectangle AGQH be completed, prove that the locus of Q is $x^3 = 2ax^2 + ay^2$.

Solution: Any normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$

This normal meets the axis $y = 0$ of the parabola in G and the tangent at the vertex i.e. $x = 0$ in H.

$\therefore$ The coordinates of G and H are $(2a+am^2, 0)$ and $(0, -2am - am^3)$ respectively.

Also the vertex A is $(0, 0)$. Let Q be (x_1, y_1).

Given that AGQH is a rectangle. AQ and GH are its diagonals and therefore there mid points are same.

$\Rightarrow$ mid point of AQ is $(1/2\ x_1, 1/2\ y_1)$ and

mid point of GH is $[1/2\ (2a+am^2), -1/2\ (2am+am^3)]$

As the mid points coincide so we have

$1/2\ x_1 = 1/2\ (2a + am^2),\ 1/2\ y_1 = -1/2\ (2am + am^3)$

or $x_1 = 2a + am^2,\ y_1 = -(2am + am^3)$

The required locus of Q is obtained by eliminating m between these.

Now $y_1^2 = (m^2a^2)(2 + m^2)^2$

$= a\ (m^2a)\ [2+am^2/a]^2$

$= a\ (x_1 - 2a)\ (x_1/a)^2$

$\Rightarrow ay_1^2 = (x_1 - 2a)\ x_1^2$

So the locus of Q is $ay^2 + 2ax^2 = x^3$

Example: Two straight lines are at right angles to one another and one of them touches $y^2 = 4a\ (x+a)$ and the other $y^2 = 4b\ (x+b)$. Prove that the point of intersection of the lines lies on the line x + a +b = 0.

Solution: We know that any tangent in terms of slope (m) of the parabola $y^2 = 4ax$ is

$y = mx + \frac{a}{m}$

Replacing x by x + a, we get

$y = m(x+a) + \frac{a}{m}$ which is tangent to $y^2 = 4a\ (x+a)$ ...(i)

Similarly, tangent in terms of slope of $y^2 = 4b\ (x+b)$ is

$y = m_1(x+b) + \frac{b}{m_1}$ ...(ii)

Given tangents are perpendicular, we have $mm_1 = -1 \text{ or } m_1 = -\frac{1}{m}$ then

(ii) becomes

$y = -\frac{(x+b)}{m} - bm$ ...(iii)

Subtracting (i) and (ii), then $0 = (x+a+b)\left(m + \frac{1}{m}\right)$

$m + \frac{1}{m} \neq 0$ Hence x + a+ b = 0

Example: If y_1, y_2, y_3 be the ordinates of a vertices of the triangle inscribed in a parabola $y^2 = 4ax,$ then show that the area of the triangle is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$.

Solution: Let $A(x_1, y_1),\ B(x_2, y_2) \text{ and } C(x_3, y_3)$ be the vertices of ΔABC. Since $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) lie on the parabola, therefore

$y_1^2 = 4ax_1, y_2^2 = 4ax_2 \text{ and } y_3^2 = 4ax_3$

$\Rightarrow \quad x_1 = \frac{y_1^2}{4a}, x_2 = \frac{y_2^2}{4a} \text{ and } x_3 = \frac{y_3^2}{4a}$

Now Area of ΔABC

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$= \frac{1}{2}\left[\frac{y_1^2}{4a}(y_2 - y_3) + \frac{y_2^2}{4a}(y_3 - y_1) + \frac{y_3^2}{4a}(y_1 - y_2)\right]$

$$= \frac{1}{8a}\left[y_1^2(y_2 - y_3) + (y_2^2 y_3 - y_2 y_3^2) - y_1\ (y_2^2 - y_3^2)\right]$$

$$= \frac{1}{8a}\left[y_1^2(y_2 - y_3) + y_2 y_3(y_2 - y_3) - y_1(y_2^2 - y_3^2)\right]$$

$$= \frac{1}{8a}(y_2 - y_3)\ \left[\mathrm{y}_1^2 + y_2 y_3 - y_1(y_2 + y_3)\right]$$

$$= \frac{1}{8a}(y_2 - y_3)\left[(y_1^2 - y_1 y_2) + (y_2 y_3 - y_1 y_3)\right]$$

$$= \frac{1}{8a}(y_2 - y_3)\left[y_1(y_1 - y_2) - y_3(y_1 - y_2)\right]$$

$$= \frac{1}{8a}(y_2 - y_3)(y_1 - y_2)(y_1 - y_3)$$

$$= -\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

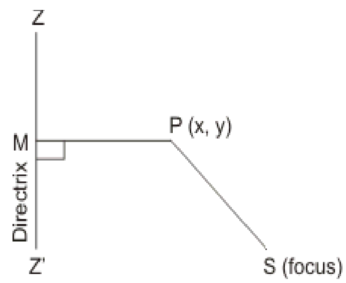
Hence, Area of ΔABC= $\frac{1}{8a}\,|\,(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\,|$

3.2 ELLIPSE

3.2.1Definition

An **ellipse** is locus of a point, which moves in a plane such that the ratio of its distance from a fixed point and a fixed line is a constant and is always less than one.

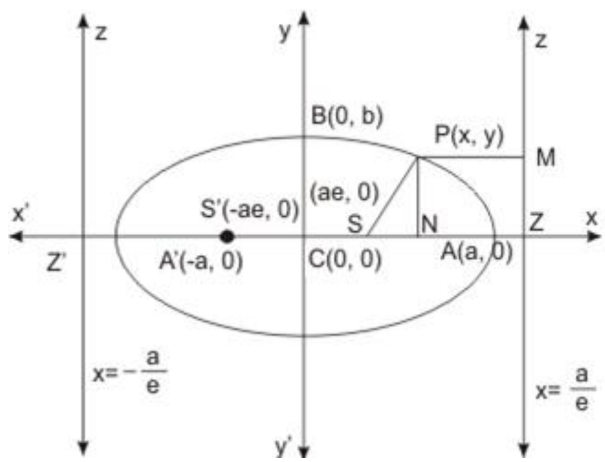
Let S be the focus, ZZ′ be the directrix and P be any point on the ellipse. Then by definition, SP/PM =e or SP=e PM, e<1, where PM is the length of the perpendicular from P on the directrix ZZ′.



Alternate definition is that an ellipse is the locus of a point that moves in such a way that the sum of its distances from two fixed points (called foci) is constant i.e., SP + S'P = constant=2a

3.2.2 Standard Equation of Ellipse

Let S be the focus and zM be the directrix of the ellipse and e be the eccentricity of the ellipse.



(This is the diagram if b<a)

Draw SZ perpendicular to the directrix and divide SZ internally and externally at A and A′ in the ratio e : 1;

Let AA' = 2a and be bisected at C.

Then SA = e AZ , SA' = e A'Z

⇒ SA + SA′ = e (AZ + ZA′) = 2ae

i.e., 2a= 2e ZC   or   ZC = a/e

⇒ SA′ – SA = e (ZA′ – ZA)

2SC = 2ae ⇒ SC = ae.

Let C be the origin, CSX be the x-axis, and the perpendicular line CY as the y-axis.

Then, S is the point (ae, 0) and ZM is the directrix, x = a/e. Let P(x, y) be any point on the ellipse.

Then by definition $SP^2 = e^2$ (Distance of P from ZM)^2

$(x - ae)^2 + y^2 = e^2 (x - a/e)^2$

or $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

i.e. $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ ... (i)

Since e < 1, $a^2 (1- e^2) < a^2 \Rightarrow b^2 < a^2$

Let $\mathbf{a^2 (1- e^2)= b^2}$.Then the equation (i) becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is the equation of ellipse in the standard form.

3.2.2.1 Important Terms

For an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

- Since only even powers of x and y occur in the equation, so the curve is symmetrical about both the axes.
- The **eccentricity e** of the ellipse is given by the relation $e^2 = (1- b^2/a^2)$.
- S and S′ are the two **foci** of the ellipse and their coordinates are (ae, 0) and (-ae, 0) respectively, then distance between foci is given by SS′=2ae.
- Corresponding to these foci, there are two **directrices** whose equations are x = a/e and x = –a/e and the distance between them is given by ZZ′=2a/e.
- The lines AA′ and BB′ are called **major axis** and **minor axis**, respectively of the ellipse.
    - The length of major axis =AA′=2a.
    - The length of minor axis= BB′=2b.
- Thee end points A(a, 0) and A′(-a, 0) of the major axis are known as the vertices of the ellipse.
- The point of intersection C of the axes of the ellipse is called the **centre** of the ellipse. All chords, passing through C, are bisected at C.
- A chord passing through its focus is called a **focal chord**.
- A focal chord, which is perpendicular to the major axis of the ellipse, is known as **latus-rectum.** The length of latus-rectum is $\frac{2b^2}{a}$. End points of the latus-rectum are $\left(\pm ae , \pm\frac{b^2}{a}\right)$.
- Let P be a point on the ellipse. From P, draw PN ⊥ AA′ (major axis of the ellipse and produce PN to meet the ellipse at P′. Then, PN is called an **ordinate** and PNP′ is called the **double ordinate** of the point P.

- The sum of the **focal distances of any point** on the ellipse is constant and is equal to its major axis. If p is any point on the ellipse, then S'P+SP = 2a.

Note: Circle is a particular case of an ellipse with e = 0.

Illustration: Find the centre, the length of the axes and the eccentricity of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$.

Solution: The given equation can be written as $2(x - 1)^2 + 3(y - 2)^2 = 1$

$(x-1)^2/(1/2)+(y-2)^2/(1/3) = 1$

$\Rightarrow$ The centre of the ellipse is (1, 2).

The major axis = 2. 1√2 = √2.

The minor axis = 2.1/√3=2/3

Eccentricity, e =1/√3 ($\because e^2 = 1 - 1/2 = 1/3$)

Illustration: Find the equation of the ellipse whose foci are (2, 3), (-2, 3) and whose semi minor axis is of length $\sqrt{5}$.

Solution: Here S is (2, 3), S' is (-2, 3) and b $=\sqrt{5}$.

$\Rightarrow$SS' = 4 = 2ae

$\Rightarrow$ ae = 2

But $b^2 = a^2 (1 - e^2)$

$\Rightarrow 5 = a^2 - 4 \Rightarrow a = 3$.

Centre C of the ellipse is (0, 3).

Hence the equation of the ellipse is $\frac{(x-0)^2}{9}+\frac{(y-3)^2}{5}=1$

$\Rightarrow 5x^2 + 9y^2 - 54y + 36 = 0$

Illustration: Find the equation of the ellipse having centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

Solution: With centre at (1, 2) the equation of the ellipse is $\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1$

It passes through the point (4, 6).

$\Rightarrow \quad \frac{\mathbf{9}}{\mathbf{a^2}}+\frac{\mathbf{16}}{\mathbf{b^2}}=\mathbf{1}$ ...... (1)

Distance between the focus and the centre = (6 - 1) = 5 = ae

$\Rightarrow \quad b^2 = a^2 - a^2e^2 = a^2 - 25.$ ...... (2)

Solving for a^2 and b^2 from the equations (1) and (2),

we get $a^2 = 45$ and $b^2 = 20$.

Hence the equation of the ellipse is $\frac{(x-1)^2}{45}+\frac{(y-2)^2}{20}=1$.

Illustration: Find the equation of the ellipse, which cuts the intercept of length 3 and 2 on positive x and y-axis. Centre of the ellipse is origin and major and minor axes are along the positive x-axis and along positive y-axis.

Solution: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...... (1)

According to the given condition the ellipse (1) passes through (3, 0) and (0, 2), so we have.

$9/a^2 = 1 \Rightarrow a^2 = 9$

And $4/b^2 = 1 \Rightarrow b^2 = 4$

Therefore, the equation of the ellipse is $\frac{\mathbf{x^2}}{\mathbf{9}} + \frac{\mathbf{y^2}}{\mathbf{4}} = \mathbf{1}$

Illustration: Obtain the equation of an ellipse whose focus is the point (-1, 1) whose directrix is the line passing through (2, 5) having the unit gradient and whose eccentricity is ½.

Solution: Let P(x, y) be any point on ellipse.

Its focus is S (-1, 1).

Let the directrix be $y = x + c$ ...... (1)

($\because$ Gradient m = 1)

Line (1) passes through (2, 5) so,

$5 = 2 + c \Rightarrow c = 3$

The directrix is $y = x + 3$

$\Rightarrow x - y + 3 = 0$ ...... (2)

Now let PM be the perpendicular from P, drawn to its directrix (2).

By definition of ellipse SP = e PM

or $SP^2 = e^2\ PM^2$

$\Rightarrow (x + 1)^2 + (y - 1)^2 = (1/2)^2\ [(x-y+3)/\sqrt{(1^2+1^2)}]^2$

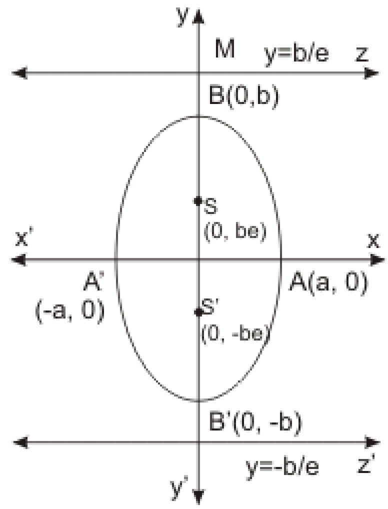
$\Rightarrow 8[(x + 1)^2 + (y - 1)^2] = (x - y + 3)^2$

$\Rightarrow 7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0,$

This is the required equation of ellipse.

3.2.2.2 Equation of ellipse in other form

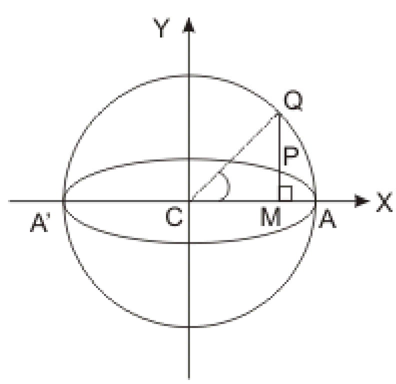
If in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a^2 < b^2$ $\Rightarrow a < b$, then the major axis of the ellipse lie along the y axis and minor axis along the x-axis.



- Coordinates of centre are (0,0)
- Coordinates of the vertices are (0 , $\pm$b)
- Coordinates of foci are (0 , $\pm$be)
- Equation of major axis is x=0 and its length is 2b
- Equation of minor axis is y=0 and its length is 2a
- Focal distances of a point (x, y) is b$\pm$ey
- Equations of directrices are $y = \pm \frac{b}{e}$
- Length of latus rectum is $\frac{2a^2}{b}$.
- Eccentricity $e = \sqrt{1 - \frac{a^2}{b^2}}$
- Parametric coordinates are (a cos θ, b sin θ)

3.2.2.3 Auxiliary circle

Auxiliary circle of an ellipse is the circle described on the major axis AA’ as its diameter.



For an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ the equation of auxiliary circle is $x^2 + y^2 = a^2$.

Let Q be a point on auxiliary circle so that QM, Perpendicular to major axis meets the ellipse at P. The points P and Q are called as corresponding points on the ellipse and auxiliary circle, respectively. The angle θ is known as eccentric angle of the point P on the ellipse. CQ is inclined at θ with x-axis.

3.2.2.4 Position of a Point Relative to an Ellipse

The point $P(x_1, y_1)$ lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0$.

The point $P(x_1, y_1)$ inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1<0$.

The point $P(x_1, y_1)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0$.

3.2.2.5 Intersection of a line and an Ellipse

The line $y= mx + c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in

- Two distinct points if $a^2m^2+b^2>c^2$.
- In one point if $a^2m^2+b^2= c^2$.
- Does not intersect if $a^2m^2+b^2<c^2$

3.2.3 Parametric Equation of the Ellipse

The coordinates $x= a \cos \theta$ and $y= b \sin \theta$ satisfy the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all real values of θ. Thus for $0\le\theta<2\pi$, $x= a \cos \theta$, $y= b \sin \theta$ are the parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The coordinates of any point on the ellipse may be taken as $(a \cos \theta, b \sin \theta)$.This point is also called the point θ. The angle θ is called the eccentric angle of the point $(a \cos \theta, b \sin \theta)$ on the ellipse.

3.2.4 Chords of Ellipse

3.2.4.1Equation of Chord

The equation of the chord joining the points $P\equiv(a\cos\theta_1, b\sin\theta_1)$ and $Q\equiv(a\cos\theta_2, b\sin\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right) .$$

Equation of a Chord Bisected at a Given Point:

The equation of a chord of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ bisected at (x_1, y_1) is given by $T = S_1$ where

$$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1 \text{ and } S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

3.2.4.2 Diameter of an Ellipse

The locus of the middle points of a system of parallel chords of an ellipse is called the diameter of the ellipse and the point where diameter intersects the ellipse is called the vertex of diameter

Equation of diameter is given by $y = -\frac{b^2}{a^2 m}x$.

3.2.4.3 Conjugate Diameters

Two diameters of an ellipse which bisects chords parallel to each other are called conjugate diameters. The eccentric angles of the ends of a pair of conjugate diameters differ by a right angle.

- If m and m_1 are the slopes of the conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $mm_1=\frac{-b^2}{a^2}$.
- Major and minor axes of an ellipse are also a pair of conjugate diameters.

3.2.5 Tangent and Normal

The condition for the line $y= mx+c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is that $c^2=a^2m^2+b^2$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}},\ m\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

3.2.5.1 Equation of Tangent in Different forms

Point Form:

The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ i.e., $T = 0$ where $T = \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.

Parametric Form:

The equation of tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a \cos \theta, b \sin \theta)$ is $\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = 1$.

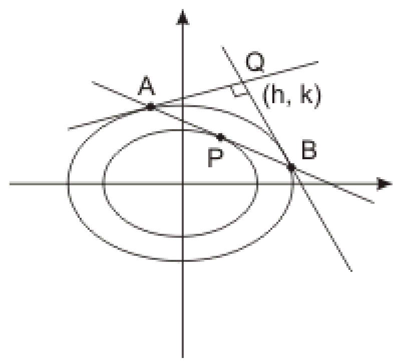
Slope Form:

The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope m is $y = mx \pm \sqrt{a^2m^2+b^2}$ and coordinates of the points of contact are $\left(\pm\dfrac{a^2m}{\sqrt{a^2m^2+b^2}},\ m\dfrac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

Note: Two tangents can be drawn from a point to a ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

Illustration: If tangent at a point P on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ intersects ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ at the point A and B. Then prove that tangents at point A and B intersect at the right angle.

Solution: Let P be ($3\cos\theta$, $2\sin\theta$) and let the tangent at point A and B to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ intersect at the point Q(h, k).



So equation of AB as a tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

$$\frac{x\cos\theta}{3}+\frac{y\sin\theta}{2}=1 \qquad \text{...(i)}$$

and equation of AB as chord of contact of the point Q with respect to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ is

$$\frac{xh}{15}+\frac{y\,k}{10}=1 \qquad \text{...(ii)}$$

As equation (i) and (ii) are representing same line.

So $\dfrac{\left(\frac{\cos\theta}{3}\right)}{\left(\frac{h}{15}\right)}=\dfrac{\left(\frac{\sin\theta}{2}\right)}{\left(\frac{k}{10}\right)}=1$

$\Rightarrow \quad \cos\theta=\frac{h}{5},\ \sin\theta=\frac{k}{5}$

$$\Rightarrow \quad \frac{h^2}{25}+\frac{k^2}{25}=1$$

So locus of (h, k) is $x^2+y^2=25=15+10$, which is the director circle of ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$. So tangent from point Q to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ will meet at right angle.

Illustration: Find the locus of the point of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) which meet at right angles.

Solution: The line $y=mx\pm\sqrt{a^2m^2+b^2}$ is a tangent to the given ellipse for all m.

Let it pass through (h, k).

$\Rightarrow$ k - mh $=\pm\sqrt{a^2m^2+b^2}$

$\Rightarrow k^2 + m^2h^2 - 2hkm = a^2m^2 + b^2$

$\Rightarrow m^2(h^2 - a^2) - 2hkm + k^2 - b^2 = 0$.

If the tangents are at right angles, then $m_1m_2 = -1$.

$\Rightarrow (k^2-b^2)/(h^2-a^2) = -1 \Rightarrow h^2 + k^2 = a^2 + b^2$.

Hence the locus of the point (h, k) is $x^2 + y^2 = a^2 + b^2$ which is a circle.

This circle is called the Director Circle of the ellipse.

Illustration: Prove that the locus of the mid-points of the intercepts of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, intercepted between the axes, is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

Solution: The tangent to the ellipse at any point (a cosθ, b sinθ) is

(x cosθ)/a+(y sinθ)/b = 1.

Let it meet the axes in P and Q, so that P is (a secθ, 0) and Q is (0, b cosec θ).

If (h, k) is the mid-point of PQ, then

h = (a secθ)/2 $\Rightarrow$ cosθ = a/2h

and k = (b cosec θ)/2 $\Rightarrow$ sinθ = b/2k.

Squaring and adding, we get $\frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$

Hence the locus of (h, k) is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

Illustration: Prove that the product of the lengths of the perpendiculars drawn from the foci to any tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is equal to 9.

Solution: For the given ellipse a = 4, b = 3 and hence $9 = 16(1 - e^2)$

$\Rightarrow e = \sqrt{7}/4$. The foci are thus located at $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$.

Equation of a tangent to the given ellipse is

$y = mx \pm \sqrt{16m^2 + 9}$ (as a = 4, b = 3).

Lengths P_1 and P_2 of the perpendiculars drawn from the foci are

$P_1 = \left|\frac{\sqrt{16m^2+9} + \sqrt{7}m}{\sqrt{1+m^2}}\right|$ and $P_2 = \left|\frac{\sqrt{16m^2+9} - \sqrt{7}m}{\sqrt{1+m^2}}\right|$

$\Rightarrow P_1P_2 = \left|\frac{16m^2+9-7m^2}{(1+m^2)}\right| = 9(1+m^2)/(1+m^2) = 9.$

Note:

The product of lengths of the perpendiculars drawn from the foci to any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is b^2.

3.2.5.2 Equation of Pair of Tangents

The Equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$SS_1 = T^2$ Where, $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, $S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ and $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$.

3.2.5.3 Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is T=0 where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Illustration: Show the locus of middle points chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which subtend right angle at the centre is $x^2/a^4 + y^2/b^4 = (1/a^2 + 1/b^2)(x^2/a^2 + y^2/b^2)^2$.

Solution: Let (x_1, y_1) be the middle point of chord PQ, then its equation is

$T = S_1$ or $xx_1/a^2 + yy_1/b^2 = (x_1/b^2)/a^2 + (y_1/b^2)/b^2$ ...... (1)

Since the origin 'O' is the centre so the equation of pair of lines OP and OQ can be obtained by homogenizing the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with the help of (1), thus

$$x^2/a^2 + y^2/b^2 = (1)^2 = \left[\frac{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}}{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}\right]^2$$

or $((x_1^2)/a^2 + (y_1^2)/b^2)^2 \, (x^2/a^2 + y^2/b^2)$

$= (x^2 x_1^2)/a^4 + (y^2 y_1^2)/b^4 + (2xyx_1 y_1)/(a^2 b^2)$.

It represents a pair of perpendicular lines if

$1/a^2 ((x_1^2)/a^2 + (y_1^2)/b^2)^2 - (x_1^2)/a^4 + 1/b^2 ((x_1^2)/a^2 + (y_1^2)/b^2)^2 - (y_1^2)/b^4 = 0$.

or, $(x_1^2)/a^4 + (y_1^2)/b^4 = (1/a^2 + 1/b^2)(x^2/a^4 + y^2/b^4)^2$.

So the locus of (x_1, y_1) is

$x^2/a^4 + y^2/b^4 = (1/a^2 + 1/b^2)(x^2/a^2 + y^2/b^2)^2$.

3.2.5.4 Director Circle

The director circle is the locus of the point of intersection of pair of perpendicular tangents to an ellipse**.**

Two perpendicular tangents of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are y - mx = $\sqrt{a^2 + b^2m^2}$ ...... (1) and

$my + x = \sqrt{a^2 + b^2m^2}$ ...... (2)

To obtain the locus of the point of intersection (1) and (2) we have to eliminate m by squaring and adding (1) and (2), we get

$(y - mx)^2 + (my + x)^2 = (a^2m^2 + b^2) + (a^2 + b^2m)$

$\Rightarrow$ $x^2 + y^2 = a^2 + b^2$, which is the equation of the director circle.

Note: The product of perpendiculars from the foci on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to b^2.

3.2.5.5 Equation of Normal in Different forms

The normal to a curve is a line perpendicular to the tangent to curve through the point of contact.

Point Form:

The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point(x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$ or

or $\dfrac{y - y_1}{\frac{y_1}{b^2}} = \dfrac{x - x_1}{\frac{x_1}{a^2}}$.

Parametric Form:

The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is ax sec θ $-$ by cosec θ =

$a^2 - b^2$ or $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.

Slope Form:

The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope m is $y = mx \pm \frac{m\left(a^2 - b^2\right)}{\sqrt{a^2 + b^2m^2}}$ and coordinates of the points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right)$.

Illustration: The tangent and normal at a point P on an ellipse meet the minor axis at A and B. Prove that AB subtends a right angle at each of foci.

Solution: The equations of the tangents and normal at a point $P(x_1, y_1)$ on the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $xx_1/a^2 + yy_1/b^2 = 1$ ...... (1)

and $(x-x_1)/((x_1/a^2))+(y- y_1)/((y_1/b^2)) = 1$ ...... (2)

Also the minor axis is y-axis i.e. $x = 0$.

Solving (1) and $x = 0$, we have A $(0, b^2/y_1)$

Solving (2) and $x = 0$, we have B $(0, y_1 - a^2y_1/b^2)$

Let S(ae, 0) be one of the foci of the ellipse. Then the slope of SA

$= ((b^2/y_1) - 0)/(0-ae)= b^2/aey_1 = m_1$ (say)

And the slope of SB $= ([y_1 (a^2/b^2) y_1]-0)/(0-ae)$

$= y_1/ae ((a^2-b^2))/b^2= (y_1 a^2 e^2)/(aeb^2)$ $[\because b^2 = a^2(1 - e^2)]$

$= aey_1/b^2 = m_2$ (say)

Evidently $m_1m_2 = -1 \Rightarrow SA \perp SB$

i.e., AB subtends a right angle at S (ae, 0). Similarly we can show that AB subtends a right angle at the other focus S'(-ae, 0)

Illustration: If the normal's to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are concurrent, prove that $\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0.$

Solution: The equation of the normal to the given ellipse at (x_1, y_1) is

$a^2xy_1 - b^2yx_1 - (a^2 - b^2)x_1y_1 = 0.$ ...... (1)

Similarly the normals at (x_2, y_2) and (x_3, y_3) are

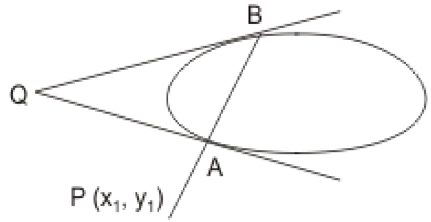
$a^2xy_2 - b^2yx_2 - (a^2 - b^2)x_2y_2 = 0.$ ...... (2)

$a^2xy_3 - b^2yx_3 - (a^2 - b^2)x_3y_3 = 0.$ ...... (3)

Eliminating a^2x, b^2y and $(a^2 - b^2)$ from (1), (2) and (3), we find that the three lines are concurrent if

$$\begin{vmatrix} y_1 & x_1 & x_1y_1 \\ y_2 & x_2 & x_2y_2 \\ y_3 & x_3 & x_3y_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0.$$

3.2.6 Pole and Polar



Let P be (x_1, y_1) and ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Tangents to the ellipse at A and B meet at Q (h, k). AB is the chord of contact with respect to Q (h, k).

$\therefore$ Equation of AB is $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$

But $P(x_1, y_1)$ lies on it, hence $\frac{hx_1}{a^2} + \frac{ky_1}{b^2} = 1$

Hence locus of Q(h, k) i.e., polar of the point P is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

Illustration: Find the pole of the line $lx + my + n = 0$ w.r.to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: Let (x_1, y_1) be the pole of line $lx + my + n = 0$. ...... (1)

w.r.to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...... (2)

Now the polar of (x_1, y_1) w.r.to (2) is

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ...... (3)

Since (1) and (3) represents the same polar, so comparing them we have

$$\frac{x_1/a^2}{l} = \frac{y_1/b^2}{m} = \frac{-1}{n}$$

or $x_1 = \frac{-a^2 l}{n}$ and $y_1 = \frac{-b^2 m}{n}$

$\therefore$ The required pole is $\left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)$

3.2.7 Solved Examples

Example: Find the equation of the ellipse (in standard form) having latus rectum 5 and eccentricity 2/3.

Solution: Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with a > b.

Latus rectum = 5 = $2b^2/a \Rightarrow 2b^2 = 5a$ ...... (1)

Also $b^2 = a^2(1 - e^2) = a^2(1-4/9) = 5a^2/9$

$\Rightarrow 5a/2 = 5a^2/9$

$\Rightarrow a = 9/2$ and

hence $b^2 = 5/2a = 45/4$.

The equation of the ellipse, in the standard form, is thus

$$\frac{x^2}{\left(\frac{81}{4}\right)} + \frac{y^2}{\left(\frac{45}{4}\right)} = 1$$

Example: If P be a point on the ellipse $x^2/a^2 + y^2/b^2 = 2/c$ whose ordinate is $\sqrt{2}/c$, prove that the angle between the tangent at P and SP is $\tan^{-1}(b^2/ac)$, where S is the focus.

Solution: The given ellipse $x^2/a^2 + y^2/b^2 = 2/c$ ...... (1)

If $(x', \sqrt{(2/c)})$ be the coordinates of the given point P on the ellipse (1).

Then the tangent at P will be: $\frac{xx'}{a^2\left(\frac{2}{c}\right)} + \frac{yy'\left(\frac{\sqrt{2}}{c}\right)}{b^2\left(\frac{2}{c}\right)} = 1$

The slope of tangent at P is $(-b^2(2/c)x')/(a(2/c)(\sqrt{2}(2/c))) = m_1$ (say)

If S be the focus, then slope of PS = $y'/(x'+ae)$

$= \sqrt{(2/c)}/(x'+a\sqrt{(2/c)}e) = m_2$(say)

If angle between the focal distance SP and tangent at P is θ, then $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

$= (a^2(2/c)+b^2x'^2 + aeb^2x'\sqrt{(2/c)})/((a^2x'+a^3e\sqrt{(2/c)}-b^2x')\sqrt{(2/c)})$

Point $(x', \sqrt{(2/c)})$ lies on ellipse (1), we have $b^2x'^2 + a^2(2/c) = (2a^2b^2)/c$ and

$\because a^2 - b^2 = a^2e^2$ so we have

$\tan\theta = (-(2a^2b^2)/c + aeb^2x'\sqrt{(2/c)})/((a^2e^2x'+a^2e\sqrt{(2/c)})\sqrt{(2/c)})$

$= (\sqrt{(2/c)}\,ab^2(\sqrt{(2/c)}\,a+ex'))/(a^2e(\sqrt{(2/c}\,a+ex'))\sqrt{(2/c)})$

$= b^2/ae$.

$\therefore \theta = \tan^{-1} b^2/ae$.

Hence angle between the tangent at P and SP is $\tan^{-1}(b^2/ac)$.

Example: If the chord joining two points whose eccentric angles are θ_1 and θ_2 cut the major axis of an ellipse at a distance d from the centre, show that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{d-a}{d+a}$, if 2a be the length of the major axis.

Solution: Equation of the chord AB joining θ_1 and θ_2 is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

It cuts the major axis at M where CM=d; hence the point M (d, 0)lies on it.

$$\therefore \frac{d}{a}=\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$$

Applying components & dividendo

$$\therefore \frac{d-a}{d+a}=\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)}=\frac{2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}{2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}$$

$$=\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$$

Example: Find the locus of the point of intersection of the two straight lines $(x \tan \alpha)/a - y/b + \tan x = 0$ and $x/a+(y \tan\alpha)/b$ where α is fixed angle. Also find the eccentric angle of the point of intersection.

Solution: Equation of the lines are given as

$(x \tan \alpha)/a-y/b + \tan \alpha = 0$ ...... (1)

$x/a+(y \tan \alpha)/b - 1 = 0$ ...... (2)

To find the locus of the point of intersection, we have to eliminate the variable '$\tan \alpha$' from (1) and (2), so by (2),

$y/b=1/(\tan \alpha)\ (1-x/a)$ and by (1)

$y/b = \tan \alpha\ (1+x/a)$

Multiplying we get

$(y/b)^2 = (1-x/a)(1+x/a)$

$\Rightarrow x^2/a^2 +y^2/b^2 = 1$

This is the equation of an ellipse

Again solving (1) and (2), we get

$x = a(1-\tan^2 \alpha)/((1+\tan^2 \alpha))$

$x = a(1-\tan^2 \alpha)/(\sec^2 \alpha)$

Let the abscissa of the point of intersection be $a \cos \theta$, then

$x = a\cos\theta = a(1-\tan^2\alpha)/(\sec^2\alpha) \quad \Rightarrow \cos\theta = (1-\tan^2\alpha)/(\sec^2\alpha)$

$\Rightarrow (1-\cos\theta)/(1+\cos\theta)=(\sec^2\alpha-(1-\tan^2\alpha))/(\sec^2\alpha+(1-\tan^2\alpha))$

(By components & dividendo)

$= (2\tan^2\alpha)/2 = \tan^2\alpha$

$\Rightarrow (2\sin^2\theta/2)/(2\cos^2\theta) = \tan^2\alpha$

$\Rightarrow \tan^2\theta/2 = \tan^2\alpha$

$\Rightarrow \tan\theta/2 = \tan\alpha$

Hence $\theta = 2\alpha$

Example: If the product of the perpendiculars from the foci upon the polar of P be constant and equal to c^2. Find the locus of P.

Solution: Suppose the equation to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The co-ordinate of foci are (ae, 0) and (-ae, 0).

Let the co-ordinates of P be (h, k). Then polar of P is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$

Or $b^2xh + a^2yk - a^2b^2 = 0$ ...... (1)

If P_1 and P_2 be the lengths of the perpendiculars on the line (1) from (ae, 0) and (-ae, 0) respectively are

$$P_1=\left|\frac{\left(b^2hae-a^2b^2\right)}{\sqrt{b^4h^2+a^4k^2}}\right| \text{ and } P_2=\left|\frac{\left(-b^2hae-a^2b^2\right)}{\sqrt{b^4h^2+a^4k^2}}\right|$$

$$\therefore P_1P_2=\frac{\left(a^4b^4-b^4h^2a^2e^2\right)}{b^4h^2+a^4k^2} = c^2 \text{ (By hypothesis)}$$

$$\Rightarrow a^4b^4-b^4h^2a^2e^2=c^2b^4h^2+c^2a^4k^2$$
$$\Rightarrow b^4h^2\left(c^2+a^2e^2\right)+c^2a^4k^2=a^4b^4$$

Generalizing the locus of the point P(h, k) is $b^4x^2\left(c^2+a^2e^2\right)+c^2a^4y^2=a^4b^4$

Example: Chords of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ always touch another concentric ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, show that the locus of their poles is $(\alpha^2x^2)/a^4 + (\beta^2y^2)/b^4 = 1$.

Solution: Let (x_1, y_1) be the pole of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...... (1)

Then the equation of this chord is the same as the polar of (x_1, y_1) w.r.to (1)

i.e., $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...... (2)

If (2) touches the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, then it should satisfy the condition of tangency (i.e., $c^2=a^2m^2+b^2$)

$\therefore$ $(b^2/y_1)^2 = \alpha^2\{-\{b^2x_1/a^2y_1\}\}^2 + \beta^2$

$\Rightarrow b^4/y_1^2 = (\alpha^2b^4x_1^2/a^4y_1^2) + \beta^2$

$\Rightarrow (\alpha^2 x_1^2)/a^4 + (\beta^2 y_1^2)/b^4 = 1$

$\therefore$ The locus of (x_1, y_1) is $(\alpha^2 x^2)/a^4 + (\beta^2 y^2)/b^4 = 1$

Hence proved.

Example: If the straight line $y = x\tan\theta + \sqrt{\left(a^2\tan^2\theta + b^2\right)/2}$, θ being the angle of inclination, intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then prove that the straight lines joining the centre to their point of intersection are conjugate diameters.

Solution: The equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...... (1)

and equation to the line is given as

$y = x\tan\theta + \sqrt{\left(a^2\tan^2\theta + b^2\right)/2}$, θ being angle of inclination

We can write this equation as, $y = mx + \sqrt{\left(a^2m^2 + b^2\right)/2}$

$$\Rightarrow \frac{(y - mx)\sqrt{2}}{\sqrt{\left(a^2m^2 + b^2\right)}} = 1 \qquad \text{...... (2)}$$

To get the equation to the lines joining the point of intersection to the origin, making (1) homogeneous with the help of (2), we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{(y - mx)\sqrt{2}}{\sqrt{\left(a^2m^2 + b^2\right)}}\right)^2 = \frac{2\left(y^2 + m^2x^2 - 2mxy\right)}{\left(a^2m^2 + b^2\right)}$$

$$\Rightarrow (b^2x^2 + a^2y^2)(a^2m^2 + b^2) = 2a^2b^2(y^2 + m^2x^2 - 2mxy)$$

$$y^2a^2(a^2m^2 - b^2) + 4m^2 - b^2xy - b^2x^2 (a^2m^2 - b^2) = 0$$

$$\Rightarrow y^2 + \left(\frac{4\,mb^2}{a^2m^2 + b^2}\right)xy - \frac{b^2}{a^2}x^2 = 0 \qquad \text{...... (3)}$$

This equation represents two straight lines $y = m_1x$ and $y = m_2x$ then the combined equation will be $y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$.

Comparing (3) and (4); we get $m_1m_2 = -b^2/a^2$

which is the condition of diameter to be conjugate. Hence the lines are the conjugate diameters.

Example: Consider the family of circles $x^2 + y^2 = r^2$, $2<r<5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the co-ordinate axes at A and B, then find the equation of the locus of the mid-points of AB.

Solution: Equation of any tangent to circle $x^2 + y^2 = r^2$ is

$x \cos\theta + y \sin\theta = r$ ...(1)

Suppose (1) is tangent to $4x^2 + 25y^2 = 100$

Or $\frac{x^2}{25} + \frac{y^2}{4} = 1$ at (x_1, y_1)

Then (1) and $\frac{xx_1}{25} + \frac{yy_1}{4} = 1$ are identical

$$\therefore \quad \frac{\frac{x_1}{25}}{\cos\theta} = \frac{\frac{y_1}{4}}{\sin\theta} = \frac{1}{r}$$

$$\Rightarrow \quad x_1 = \frac{25\cos\theta}{r}, \; y_1 = \frac{4\sin\theta}{r}$$

The line (1) meets the coordinate axes in

A $(r\sec\theta, 0)$ and B$(0, r\,\mathrm{cosec}\theta)$, Let (h, k) be mid point of AB.

Then $h = \frac{r\sec\theta}{2}$ and $k = \frac{r\,\mathrm{cosec}\theta}{2}$

Therefore, $2h = \frac{r}{\cos\theta}$ and $2k = \frac{r}{\sin\theta}$

$\therefore \; x_1 = \frac{25}{2h}$ and $y_1 = \frac{4}{2k}$

As (x_1, y_1) lies on the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$

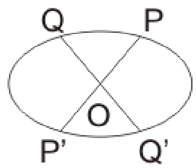
we get $\frac{1}{25}\left(\frac{625}{4h^2}\right) + \frac{1}{4}\left(\frac{4}{k^2}\right) = 1$

$$\Rightarrow \quad \frac{25}{4h^2} + \frac{1}{k^2} = 1$$

Or $25k^2 + 4h^2 = 4h^2k^2$

Therefore, required locus is $4x^2 + 25y^2 = 4x^2 y^2$

Example: Prove that the sum of the squares of the reciprocals of two perpendicular diameters of an ellipse is constant.



Solution: Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ …(1)

And PP′, QQ′ be two perpendicular diameters.
Let the equation of PP′ be y = mx …(2)
Solving (1) and (2).

$$x^2\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) = 1$$

$$\therefore \quad x = \pm \frac{ab}{\sqrt{a^2m^2 + b^2}}$$

$$\therefore \quad y = \frac{\pm mab}{\sqrt{a^2m^2 + b^2}}$$

$$\therefore \ PP'^2 = (2 \times OP)^2 = 4\left\{\frac{a^2b^2}{a^2m^2 + b^2} + \frac{m^2a^2b^2}{a^2m^2 + b^2}\right\}$$

$$= \frac{4a^2b^2}{a^2m^2 + b^2}(1 + m^2).$$

As $QQ' \perp PP'$ and QQ' passes through the centre O(0, 0), the equation of QQ' will be

$$y = \frac{-1}{m}x \quad \text{...(3)}$$

Solving (1), (3) and proceeding as above,

$$QQ'^2 = \frac{4a^2b^2}{a^2\left(-\frac{1}{m}\right)^2 + b^2}\left\{1 + \left(-\frac{1}{m}\right)^2\right\}$$

$$= \frac{4a^2b^2m^2}{a^2 + b^2m^2} \cdot \frac{1 + m^2}{m^2}$$

$$= \frac{4a^2b^2}{a^2 + b^2m^2}(1 + m^2)$$

$$\therefore \quad \frac{1}{PP'^2} + \frac{1}{QQ'^2} = \frac{a^2m^2 + b^2}{4a^2b^2(1 + m^2)} + \frac{a^2 + b^2m^2}{4a^2b^2(1 + m^2)}$$

$$= \frac{1}{4a^2b^2(1 + m^2)}(a^2m^2 + b^2 + a^2 + b^2m^2)$$

$$= \frac{(a^2 + b^2)(1 + m^2)}{4a^2b^2(1 + m^2)} = \frac{a^2 + b^2}{4a^2b^2} = \text{constant.}$$

Example: If the line 3y = 3x + 1 is a normal to the ellipse $\frac{x^2}{5}+\frac{y^2}{b^2}=1$, then find out the length of the minor axis of the ellipse.

Solution: Equation of normal with slope m to the given ellipse is

$$y = mx \pm \frac{m(5-b^2)}{\sqrt{b^2m^2+5}}$$

$$\Rightarrow \quad y = x \pm \frac{(5-b^2)}{\sqrt{b^2+5}} \quad (\text{as } m = 1) \qquad ...(i)$$

$$y = x + \frac{1}{3} \qquad ...(ii)$$

both (i) and (ii) represents same line

$$\text{so} \quad \pm\frac{(5-b^2)}{\sqrt{b^2+5}} = \frac{1}{3}$$

$$\Rightarrow \quad (5-b^2)^2 = \frac{1}{9}(5+b^2)$$

$$\Rightarrow \quad 9b^4 - 91b^2 + 220 = 0$$

$$\Rightarrow \quad b^2 = 4 \text{ or } 55/9$$

so length of the minor axis is $4 \text{ or } \frac{2}{3}\sqrt{55}$.

Example: The tangents drawn from a point P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make angles θ_1 and θ_2 with the major axis; find locus of P when $\tan^2\theta_1 + \tan^2\theta_2 = \lambda$ (a constant).

Solution: Any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = mx \pm \sqrt{a^2m^2+b^2}$

Let P = (h, k)

$\Rightarrow$ $k = mh \pm \sqrt{a^2m^2+b^2}$

$\Rightarrow$ $(k - mh)^2 = a^2m^2 + b^2$

$\Rightarrow$ $m^2(h^2 - a^2) - 2hkm + (k^2 - b^2) = 0$ ...(i)

$\Rightarrow$ $m_1 + m_2 = \frac{2hk}{h^2-a^2} \text{ and } m_1m_2 = \frac{k^2-b^2}{h^2-a^2}$ ...(ii)

If θ_1 and θ_2 are the angles of inclination of tangents to the x-axis, then

$$\tan\theta_1 + \tan\theta_2 = \frac{2hk}{h^2-a^2} \text{ and } \tan\theta_1\tan\theta_2 = \frac{k^2-b^2}{h^2-a^2}$$

Given that $\tan^2\theta_1 + \tan^2\theta_2 = \lambda$

$\Rightarrow$ $(\tan\theta_1 + \tan\theta_2)^2 - 2\tan\theta_1\tan\theta_2 = \lambda$

$\Rightarrow$ $\left(\frac{2hk}{h^2-a^2}\right)^2 - \frac{2(k^2-b^2)}{h^2-a^2} = \lambda$ (from (ii))

$\Rightarrow$ $4h^2k^2 - 2(k^2 - b^2)(h^2 - a^2) = \lambda(h^2 - a^2)^2$

$\Rightarrow$ $2(h^2k^2 + k^2a^2 + h^2b^2 - a^2b^2) = \lambda(h^2 - a^2)^2$

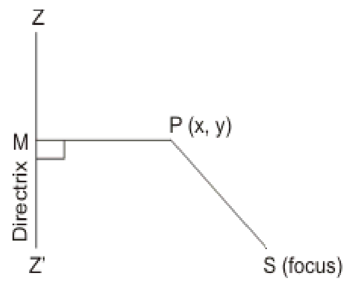
$\Rightarrow$ Required locus is $2(x^2y^2 + a^2y^2 + x^2b^2 - a^2b^2) = \lambda(x^2 - a^2)^2$

3.3 Hyperbola

3.3.1 Definition

A hyperbola is the locus of a point which moves in a plane so that the ratio of its distances from a fixed point (called the focus) and a fixed line (called directrix) is a constant which is greater than one. This ratio is called eccentricity and is denoted by e.

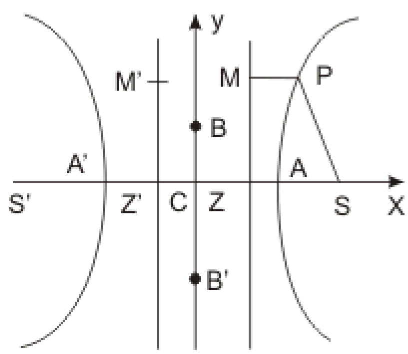
Let S be the focus, ZZ′ be the directrix and P be any point on the hyperbola. Then by definition, SP/PM =e or SP=e PM, e>1, where PM is the length of the perpendicular from P on the directrix ZZ′.



or A hyperbola is the locus of a point which moves in such a way that the difference of its distances from two fixed points(called foci) is constant. i.e., SP - S'P = constant=2a

3.3.2 Standard Equation of hyperbola

Let S be the focus and ZM be the directrix of a hyperbola.



Draw SZ perpendicular from S on the directrix ZM. Divide SZ internally and externally at A and A′ in the ratio e: 1;

Let AA′ = 2a and be bisected at C.

Then, SA = e.AZ, SA′ = e.ZA′.

⇒ SA + SA′ = e (AZ + ZA′) = 2ae

i.e., 2SC = 2ae or SC = ae.

⇒ SA′ − SA = e (ZA′ − ZA)

AA′= 2e.ZC

⇒ 2a = 2eSC ⇒ CZ = a/e.

Let C be the origin, CSX be the x-axis, and the perpendicular line CY as the y-axis.

Then, S is the point (ae, 0) and ZM (directrix) is the line x = a/e. Let P(x, y) be any point on the hyperbola.

Then by definition $SP^2 = e^2$ (Distance of P from ZM)^2

$(x - ae)^2 + y^2 = e^2 (x - a/e)^2$

or $x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$

i.e. $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$ ... (i)

Since e>1, $e^2 - 1$ is positive.

Let $\mathbf{a^2(e^2 - 1) = b^2}$.

Then the equation (i) becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which is the equation of hyperbola in the standard form.

Illustration: Find the equation of hyperbola whose foci are (2, 4) and (10, 4) & eccentricity is 2.

Solution: We know that the centre of hyperbola is the mid point of two foci

i.e., coordinates of centres are (6, 4).

We also know that the distance between two foci is 2ae.

i.e., $(10 - 2)^2 + (4 - 4)^2 = (2ae)^2$

$\Rightarrow 2ae = 8$

$\Rightarrow 4a = 8$

$\Rightarrow a = 2$

$b^2 = a^2 (e^2 - 1)$,

Hence $b^2 = 4(4 - 1) = 12$

The required equation of hyperbola is $\frac{(x-6)^2}{4} - \frac{(y-4)^2}{12} = 1$

3.3.2.1 Important Terms

- The **eccentricity e** of the hyperbola is given by the relation $e^2 = (1 + b^2/a^2)$.
- Since only even powers of x and y occur in the equation, so the curve is symmetrical about both the axes.
- S and S′ are the two **foci** of the hyperbola and their coordinates are (ae, 0) and (-ae, 0) respectively, then distance between foci is given by SS′=2ae.
- Corresponding to these foci, there are two **directrices** whose equations are x = a/e and x = −a/e.
- The points A and A′ where the straight line joining the two foci cuts the hyperbola are called the **vertices** of the hyperbola.
- The point of intersection C of the axes of the hyperbola is called the **centre** of the hyperbola. All chords, passing through C, are bisected at C.
- The lines AA′ and BB′ are called **transverse axis** and **conjugate axis**, respectively of the hyperbola.
    - The length of transverse axis=AA′=2a.

- The length of Conjugate axis= BB'=2b.

- A chord passing through its focus is called a **focal chord**.
- The difference of the **focal distances of any point** on the hyperbola is constant and equal to the length of the transverse axis of the hyperbola. If p is any point on the hyperbola, then S'P-SP = 2a = Transverse axis.
- A **latus rectum** is the chord through a focus at right angle to the transverse axis. The length of the **semi-latus rectum** can be obtained by putting x = ae in the equation of the hyperbola. Thus $y = b\sqrt{\frac{a^2e^2}{a^2} - 1} = b\sqrt{e^2 - 1} = b \times \frac{b}{a} = \frac{b^2}{a}$.Hence the length of latus rectum is $2\frac{b^2}{a}$.

Illustration: Find the directrix, foci and eccentricity of the hyperbola $ax^2 - y^2 = 1$

Solution: The given hyperbola is $ax^2 - y^2 = 1$

or $$\frac{x^2}{\left(1/a\right)} - \frac{y^2}{1} = 1 \qquad \text{...... (1)}$$

which is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Here $a^2 = 1/a$, $b^2 = 1$.

If e be the eccentricity of the hyperbola, then $b^2 = a^2(e^2 - 1)$

$\Rightarrow 1 = 1/a\ (e^2 - 1)$

$\Rightarrow a = (e^2 - 1)$

or $e^2 = a + 1$ or $e = \sqrt{a+1}$

Also the foci are given by $(\pm\ ae, 0)$

$\therefore$ The required foci are $\left(\pm\frac{\sqrt{a+1}}{\sqrt{a}}\ ,\ 0\right)$

And the directrices are given by $x = \pm\ (a/e)$

$\Rightarrow\ x = \pm\frac{1}{\sqrt{a}\sqrt{a+1}}$

Illustration: Find the equation of the hyperbola the distance between whose foci is 16, whose eccentricity is √2 and whose axis is along the x-axis centre being the origin.

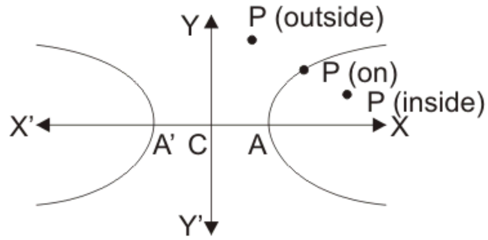
Solution: We have $b^2 = a^2(e^2 - 1) = a^2 \qquad \Rightarrow b = a$.

Also $2ae = 16 \Rightarrow ae = 8 \Rightarrow a = 4\sqrt{2}$.

Hence the equation of the required hyperbola is

$x^2/32 - y^2/32 = 1 \Rightarrow x^2 - y^2 = 32$.

3.3.2.2 Position of a Point Relative to a Hyperbola



The point P(x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0, = 0 \text{ or } < 0.$

3.3.2.3 Intersection of a line and Hyperbola

The line y = mx + c meets the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in real, coincident or imaginary points depending up on whether is c^2 greater, equal or less than $a^2m^2 - b^2$.

Illustration: Show that the line 4x – 3y = 9 touches the hyperbola $4x^2 - 9y^2 = 27$.

Solution: We know that if the line y = mx + c touches the hyperbola $x^2/a^2 - y^2/b^2 = 1$, then $c^2 = a^2m^2 - b^2$

Here the hyperbola is $\frac{x^2}{\left(27/4\right)} - \frac{y^2}{\left(27/9\right)} = 1$

i.e., here $a^2 = 27/4$, $b^2 = 27/9 = 3$

And comparing 4x – 3y = 9 with y = mx + c, we get

'm' = 4/3, 'c' = –3

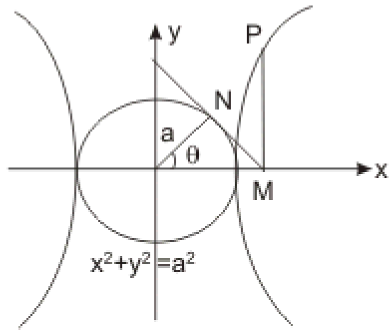
$\therefore a^2m^2 - b^2 = (27/4)(4/3)^2 - 3 = 12 - 3 = 9 = (-3)^2$

or $a^2m^2 - b^2 = c^2$

Hence the given line touches the given hyperbola.

3.3.3 Parametric Equation of Hyperbola

We can express the coordinate of a point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of a single parameter, say θ.



In the above figure OM = a sec θ and PM = b tan θ. Therefore, x = a sec θ, y = b tan θ are the parametric equations of the hyperbola $\frac{x^2}{a^2}$ - $\frac{y^2}{b^2} = 1$. Hence, the coordinates of any point on the hyperbola may be taken as (a sec θ, b tan θ) .This point is also called the point **θ**.

The angle θ is called the eccentric angle of the point (a sec θ, b tan θ) on the hyperbola.

3.3.4 Conjugate Hyperbola

A hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola. The asymptotes of these two hyperbolas are also the same.

Equation of a conjugate hyperbola is $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ ... (1) of the given hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.Its transverse and conjugate axes are along y and x axes respectively.

Note:

- Any point on conjugate hyperbola (1) is (a tan θ, b sec θ)
- The equation of the conjugate hyperbola to $xy = c^2$ is $xy = -c^2$.

Illustration: If e_1 and e_2 are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola respectively, prove that $e_1^{-2} + e_2^{-2} = 1$.

Solution: The eccentricity e_1 of the given hyperbola is obtained from

$b^2 = a^2 (e_1^2 - 1)$. ...... (1)

The eccentricity e_2 of the conjugate hyperbola is given by

$a^2 = b^2 (e_2^2 - 1)$. ...... (2)

Multiply (1) and (2), we get,

$1 = (e_1^2 - 1)(e_2^2 - 1) = e_1^2 e_2^2 - e_1^2 - e_2^2$

$\Rightarrow e_1^{-2} + e_2^{-2} = 1$.

Illustration: Find the length of transverse axis, conjugate axis eccentricity, coordinates of foci, vertices, length of latus rectum & equation of directrices of the hyperbola $9x^2 - 4y^2 = -36$.

Solution: The hyperbola is $9x^2 - 4y^2 = -36$

$\Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = -1$ i.e., of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ where a = 2 and b = 3

Length of transverse axis = 2b = 6units
Length of conjugate axis = 2a = 4 units

Eccentricity = $\sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{4}{9}} = \frac{\sqrt{13}}{3}$

Vertices = (0, ± b) = (0, ± 3)

Length of Latus Rectum = $\frac{2a^2}{b} = \frac{8}{3}$ units

Equation of directrices = y = ± b/e

$\Rightarrow y = \pm \frac{3}{\sqrt{13}/3} \quad \Rightarrow y = \pm \frac{9}{\sqrt{13}}$

$\Rightarrow \; y = \pm \frac{9\sqrt{13}}{13}$

3.3.5 Chords of Hyperbola

3.3.5.1 Equation of Chord

The equation of the chord joining the points $P \equiv (a\sec\theta_1, b\tan\theta_1)$ and $Q \equiv (a\sec\theta_2, b\tan\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right) \text{ or } \begin{vmatrix} x & y & 1 \\ a\sec\theta_1 & a\tan\theta_1 & 1 \\ a\sec\theta_2 & a\tan\theta_2 & 1 \end{vmatrix}$$

Equation of a Chord Bisected at a Given Point:

The equation of a chord of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ bisected at (x_1, y_1) is given by $T = S_1$ where

$$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 \text{ and } S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

3.3.5.2 Diameter of Hyperbola

The locus of the middle points of any set of parallel chords of a hyperbola is called a diameter and the point where diameter intersects the hyperbola is called the vertex of diameter and equation of diameter is $y = \frac{b^2x}{a^2m}$.

3.3.5.3 Conjugate Diameters

Two diameters of a hyperbola are said to be conjugate diameters if each bisects the chords parallel to the other. If m_1 and m_2 be the slopes of the conjugate diameters of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $m_1 m_2 = \frac{b^2}{a^2}$.

3.3.6 Tangent and Normal

The condition for the line y=mx+c to be tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $c^2=a^2m^2-b^2$ and the coordinates of the points of contact are $\left(\pm \frac{a^2 m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 - b^2}} \right)$

3.3.6.1 Equation of Tangent in Different forms

Point Form:

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(x_1 y_1)$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ i.e., T = 0 where

$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Parametric Form:

The equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$ is $\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$.

Slope Form:

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$ and coordinates of points of contact are $\left(\pm \frac{a^2 m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 - b^2}} \right)$

Note: Two tangents can be drawn from a point to a hyperbola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the hyperbola.

3.3.6.2 Equation of Pair of Tangents

The Equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$SS_1=T^2$ Where, $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ and $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

3.3.6.3 Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is T=0 where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

3.3.6.4 Director Circle

The director circle is the locus of the point where perpendicular tangents of the hyperbola meet.

Equation of the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$ i.e. a circle whose centre is origin and radius is $\sqrt{(a^2 - b^2)}$.

Note:

- If $b^2 < a^2$, this circle is real.
- If $b^2 = a^2$, the radius of the circle is zero, and it reduces to a point circle at the origin. In this case the centre is only point from where tangents at right angle can be drawn to the hyperbola.
- If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle, and so no tangents at right angles can be drawn to the circles.

3.3.6.5 Equation of Normal in Different forms

Point Form:

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1,y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 + b^2$

Parametric Form:

The equation of normal at (a $\sec\theta$, $\tan\theta$) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$ax \cos\theta + by \cot\theta = a^2 + b^2$ or $\frac{ax}{\sec\theta} - \frac{by}{\tan\theta} = a^2 + b^2$

Slope Form:

The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope m is

$y = mx \pm \frac{m\left(a^2 + b^2\right)}{\sqrt{a^2 - b^2m^2}}$ and coordinates of points of contact are $\left(\pm \frac{a^2}{\sqrt{a^2 - b^2m^2}},\ m\frac{mb^2}{\sqrt{a^2 - b^2m^2}}\right)$

Illustration: Write down the equation of the tangent and normal at the point (4, $3\sqrt{3}$) to the hyperbola $9x^2 - 4y^2 = 36$.

Solution: The equation of the Hyperbola can be written in the form:-

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

So the equation of the tangent at the required point is:-

$$\frac{4x}{4} - \frac{3\sqrt{3}y}{9} = 1 \text{ or } 3x - \sqrt{3}y = 3$$

And the equation of the Normal at the same point is:-

$$\frac{x-4}{\frac{4}{4}} - \frac{y-3\sqrt{3}}{\frac{3\sqrt{3}}{-9}} \quad \text{or} \quad x + 3\sqrt{3}y = 3$$

3.3.7 Pole and Polar

Let P be (x_1, y_1) and hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If tangents to the hyperbola at A and B meets at Q(h, k), then AB is the chord of contact with respect to Q(h,k).

$\therefore$ Equation of AB is $\frac{hx}{a^2} - \frac{ky}{b^2} = 1$

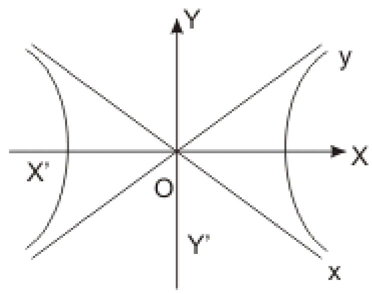
But P(x_1, y_1) lies on it, hence $\frac{hx_1}{a^2} - \frac{ky_1}{b^2} = 1$

Hence locus of Q (h, k) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

This is required equation of polar with (x_1, y_1) as its pole.

3.3.8 Asymptotes

An asymptote of a hyperbola or any curve is a straight line which touches the curve at infinity at two points.



To find the equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

Let y = mx + c be an asymptotic to the given hyperbola.

Then by eliminating y, the abscissae of the points of intersection of y = mx + c

or $x^2(b^2 - a^2m^2) - 2a^2mcx - a^2(b^2 + c^2) = 0$ …… (1)

If the line y = mx + c is an asymptote of the hyperbola then it touches the hyperbola at infinity i.e. both the roots of the equation (1) are infinite.

For this we must have $b^2 - a^2m^2 = 0$ and $2a^2mc = 0$.

Hence we get $m = \pm\frac{b}{a}$ and c = 0.

$\therefore$ **The asymptotes are** $y = \pm\frac{b}{a}x$ or $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 0$

Note:

- The angle between the asymptotes is 2 $\tan^{-1}$ (b/a).
- The equation of the asymptote differs from that of the hyperbola in the constant term only.
- The lines $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 0$ are also asymptotes to the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- Any line drawn parallel to the asymptote of the hyperbola would meet the curve only at one point.
- The asymptotes pass through the centre of the hyperbola.
- The product of the perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is a constant equal to $\frac{a^2b^2}{a^2+b^2}$

Illustration: Find the hyperbola whose asymptotes are 2x – y = 3 and 3x + y – 7 = 0 and which passes through the point (1, 1).

Solution: The equation of the hyperbola differs from the equation of the asymptotes by a constant.

The equation of the hyperbola with asymptotes 3x + y – 7 = 0 and 2x – y = 3 is (3x + y – 7) (2x – y – 3) + k = 0.

It passes through (1, 1) ⇒ k = –6.

Hence the equation of the hyperbola is (2x – y – 3) (3x + y – 7) = 6.

Illustration: Find the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose length of latus rectum is 4/3 and which passes through the point (4, 2).

Solution: We have $2b^2/a$ = length of the latus rectum = 4/3 ⇒ $3b^2 = 2a$

$$\Rightarrow \ b^2 = \frac{2}{3}a$$

Also, the hyperbola passes through the point (4, 2).

Hence $\frac{16}{a^2} - \frac{4}{b^2} = 1 \quad \Rightarrow \frac{16}{a^2} - \frac{6}{a} = 1$.

or $a^2 + 6a - 16 = 0 \Rightarrow (a - 2)(a + 8) = 0$

⇒ a = 2

⇒ $b^2 = 4/3$.

The asymptotes of the given hyperbola are $y = \pm \frac{b}{a}x$ or $y = \pm \frac{1}{\sqrt{3}}x$.

If θ_1 and θ_2 are the angles which the asymptotes make with the positive x-axis, then

$\tan\theta_1 = \frac{1}{\sqrt{3}} \Rightarrow \theta_1 = \frac{\pi}{6}$ and $\tan\theta_2 = -\frac{1}{\sqrt{3}} \Rightarrow \theta_2 = -\frac{\pi}{6}$.

Hence the angle between the asymptotes = $\frac{\pi}{3}$.

Illustration: Prove that the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which touch its conjugate hyperbola are bisected at the point of contact.

Solution: Any tangent to the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is

$x = my + \sqrt{b^2m^2 - a^2}$. ...... (2)

If this is same as the chord (1), then $m = \frac{a^2y_1}{b^2x_1}$ and hence

$$\frac{a^4}{{x_1}^2}\left[\frac{{x_1}^2}{{a_1}^2} - \frac{{y_1}^2}{{b_1}^2}\right]^2 = b^2\left(\frac{a^4{y_1}^2}{b^4{x_1}^2}\right) - a^2$$

$$\left[\frac{{x_1}^2}{{a_1}^2} - \frac{{y_1}^2}{{b_1}^2}\right]^2 = \frac{{y_1}^2}{{b_1}^2} - \frac{{x_1}^2}{{a_1}^2} \text{ or } \frac{{x_1}^2}{{a_1}^2} - \frac{{y_1}^2}{{b_1}^2} = -1$$

$\Rightarrow$ (x_1, y_1) lies on the conjugate hyperbola.

$\Rightarrow$ the chord (1) touches conjugate hyperbola and is bisected at the point of contact.

3.3.9 Rectangular Hyperbola

A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.

The angle between asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is $2\tan^{-1}\left(\frac{b}{a}\right)$

This is a right angle if $\tan^{-1}\frac{b}{a} = \frac{\pi}{4}$, i.e., if $\frac{b}{a} = 1 \Rightarrow b = a$.

The general equation of the rectangular hyperbola is $x^2 - y^2 = a^2$.

- The equations of asymptotes of the rectangular hyperbola are $y = \pm x$.
- The transverse and conjugate axes of a rectangular hyperbola are equal in length.
- Eccentricity , $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{2}$

Equation of the rectangular hyperbola referred to asymptotes as coordinate axes is $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

- The parametric coordinates are $x = ct$ and $y = c/t$.
- Equation of chord joining t_1 and t_2 is $\mathbf{x + y\, t_1t_2 = c(t_1 + t_2)}$
- Equation of tangent at t is $\frac{x}{t} + yt = 2c$

- Point of intersection of tangents at 't_1' & 't_2' is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1t_2}\right)$.
- Equation of normal at 't' is $xt^3 - yt - ct^4 + c = 0$.
- Tangent drawn at any point to $xy=c^2$ would always make an obtuse angle with the x-axis.
- Normal drawn at any point to $xy=c^2$ would always make an acute angle with the x-axis.

Illustration: A circle and a rectangular hyperbola meet in four points A, B, C and D. If the line AB passes through the centre of the circle, prove that the centre of the hyperbola lies at the mid-point of CD.

Solution: The line AB passes through the centre of the circle.

Hence AB is the diameter of the circle and the mid-point of AB is the centre of the circle.

Let the co-ordinates of A, B, C, D be respectively (x_1, y_1) (x_2, y_2), (x_3, y_3) and (x_4, y_4).

Let the centres of the hyperbola and the circle be (h, k) and (g, f).

Then $x_1+x_2x_3+x_4/4 = h+g/2$. But $g = x_1+x_2/2$

$\Rightarrow 2g+x_3/x_4/4 \Rightarrow h+g/2 \Rightarrow x_3+x_4/2 = h$

Similarly $y_3+y_4/2 = k$.

Hence (h, k) is the mid-point of CD.

3.3.10 Solved Examples

Example: Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus is (1, 1) and eccentricity is $\sqrt{3}$.

Solution: Let S (1, 1) be focus and P(x, y) be any point on the hyperbola.

From P draw PM perpendicular to the directrix then $PM = 2x+y-1/\sqrt{2^2+1^2}$

$= 2x+y-1/\sqrt{5}$

Also from the definition of the hyperbola, we have

$SP/PM = e \Rightarrow SP = e\,PM$

$\Rightarrow \sqrt{(x-1)^2+(y-1)^2} = \sqrt{3}\,(2x+y-1/\sqrt{5})$

$\Rightarrow (x-1)^2 + (y-1)^2 = 3\,(2x+y-1)^2/5$

$\Rightarrow 5[(x^2+1-2x)+(y^2+1-2y)]=3(4x^2+y^2+1+4xy-4x-2y)$

$\Rightarrow 7x^2 - 2y^2 + 12xy - 2x - 4y - 7 = 0$

Example: From a point A, perpendiculars AB and AC are drawn to two straight lines OB and OC. If the area OBAC is constant, find the locus of A.

Solution: Let the bisectors of the angles BOC be taken as axis. So the equations of OB and OC are respectively.

$x \cos\alpha + y \sin\alpha = 0$

and $x \cos\alpha - y \sin\alpha = 0$ where $\alpha = 1/2\,\angle BOC$

Take any point A as (h, k); then

AB = Perpendicular distance from A on OB

$= h \cos\alpha + k \sin\alpha / \sqrt{\cos^2\alpha + \sin^2\alpha} = h \cos \alpha + k \sin \alpha$ ...... (1)

and similarly AC = Perpendicular from A on C

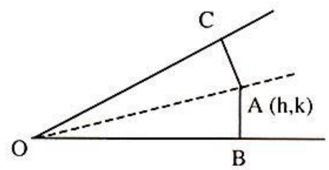
$= h \cos \alpha - k \sin \alpha$ ...... (2)

The equation to AB will be $(h - x) \sin \alpha + (y - k) \cos \alpha = 0$

$\Rightarrow y \cos\alpha - x \sin\alpha + h \sin \alpha - k \cos \alpha = 0$ ...... (3)

Similarly the equation AC will be

$(h - x) \sin \alpha - (y - k) \cos \alpha = 0$



Now OB = Perpendicular distance of (3) from (0, 0).

$= 0-0+h \cos\alpha + k \sin\alpha / \sqrt{\cos^2\alpha + \sin^2\alpha}$

$= h \sin \alpha - k \cos \alpha$

Similarly OC = perpendicular distance of (0, 0) from (4)

$= h \sin\alpha + k \cos\alpha$.

Now the area of quadrilateral OBAC = $\Delta OAB + \Delta OAC$

$= 1/2\ OB \times AB + 1/2\ OC \times AC$

$= 1/2 [h \sin\alpha - k\cos\alpha][h \cos\alpha + k \sin\alpha] + 1/2 [h \cos\alpha - k \sin\alpha] [h \sin\alpha + k \cos\alpha]$

$= (h^2 - k^2) \sin\alpha.\cos\alpha$ = constant = S (say)

$\Rightarrow h^2 - k^2 = \{s/\sin\alpha, \cos\alpha\}$ which is again constant $= a^2$ (say)

Therefore the locus of the point (h, k) will be $x^2 - y^2 = a^2$, which is hyperbola.

Example: Prove that the angle subtended by any chord of a rectangular hyperbola at the centre is the supplement of the angle between the tangents at the end of the chord.

Solution: Let the equation of the hyperbola be $x^2 - y^2 = a^2$ and P and Q be any two points on it such that their coordinates are respectively $(a \sec \theta_1, a \tan \theta_1)$ and $(a \sec \theta_2, a \tan \theta_2)$ and C(0,0) be the centre of the hyperbola.

Equation of the line PC is $y - 0 = a \tan\theta_1 - 0 / a \sec\theta_1 - 0\ (x - 0)$

$\Rightarrow y = x \sin \theta_1$ ...... (1)

Similarly equation to QC will be $y = x \sin \theta_2$ ...... (2)

If α be the angle between PC and QC, then

$\tan \alpha = \sin\theta_1 - \sin\theta_2 / 1 + \sin\theta_1 \sin\theta_2$ ...... (3)

Again the equation to the tangent at P is

$x \, a \sec \theta_1 - y \, a \tan \theta_1 = a^2$

$y = x/\sin\theta_1 - a\cos\theta_1/\sin\theta_1$ …… (4)

Similarly the equation to the tangent at Q is

$y = x/\sin\theta_2 - a \cos\theta_2/\sin\theta_2$

If β be the angle between the tangents at the end of the chord, then

$\tan \beta = 1/\sin\theta_2 \; 1/\sin\theta_2/1+1/\sin\theta_1 \; 1/\sin\theta_2 = \sin\theta_2-\sin\theta_1/1+\sin\theta_1\sin\theta_2$

$= -(\sin\theta_1-\sin\theta_2)/1+(\sin\theta_1\sin\theta_2)$

$\Rightarrow \tan\beta = -\tan\alpha$

$\Rightarrow \tan\beta = \tan(\pi - \alpha)$ (By (3))

$\Rightarrow \beta = \pi - \alpha$ Hence proved.

Example: Prove that the locus of the pole of a chord of the hyperbola which subtends a right angles at the vertex, is, $x = a^2-b^2/a^2+b^2$.

Solution: The coordinates of the vertex are (a, 0). Transferring the origin to this point, the equation of the hyperbola $x^2/a^2-y^2/b^2$ becomes;

$(x+a)^2/a^2 - y^2/b^2 = 1$

$\Rightarrow x^2/a^2-y^2/b^2 = -2x/a$

$\Rightarrow b^2x^2 - a^2y^2 = -2ab^2x$ …… (1)

The equation to the polar of (h, k) w.r.to hyperbola is given by

$b^2hx - a^2ky = a^2b^2$ …… (2)

After transformation the equation (2) becomes

$b^2(x + a)h - a^2yk = a^2b^2$

or $b^2hx - a^2yk = a^2b^2 - ab^2h$ …… (3)

The equation of the lines joining the points of intersection of the hyperbola and the chord to the origin is obtained by making (1) homogeneous with the help of (3). Hence on simplification, this equation becomes

$(a^2b^2 - ab^2h)(b^2x^2 - a^2y^2) = -2ab^2x(b^2xh - a^2yk)$

If they are at right angles, the sum of the coefficients of x^2 and y^2 must be zero; hence

$b^2 - a^2 + 2b^2h/a-h = 0$

Generalizing for (h, k), we get the required locus as $x = a^2-b^2/a^2+b^2$

Example: Find the locus of intersection of tangent to a hyperbola, which meet at a constant angle β.

Solution: Let the equation to the hyperbola be $x^2/a^2-y^2/b^2 = 1$ …… (1)

Equation to any tangent to (1) is $y = mx + \sqrt{a^2m^2-b^2}$

If the tangent passes through a point (h, k) when we must have

$k = mh + \sqrt{(a^2m^2-b^2)}$

or $m^2(h^2 - a^2) - 2mhk + (k^2 + b^2) = 0$ …… (2)

Let m_1 and m_2 be the two roots of this equation.

$\Rightarrow m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, we have

$\tan\theta_1 + \tan\theta_2 = 2hk/h^2-a^2$

$\tan\theta_1 \times \tan\theta_2 = k^2+b^2/h^2-a^2$

and as $(\tan\theta_1 - \tan\theta_2)^2 = 4h^2k^2-4(k^2+b^2)(h^2-a^2)/(h^2-a^2)^2$.

$= 4(a^2k^2-b^2h^2+a^2b^2)/(h^2-a^2)^2$

If the two tangents met at an angle β, clearly $\beta = (\theta_1 - \theta_2)$.

Hence $\cot\beta = \cot(\theta_1 - \theta_2)$

$= 1/\tan(\theta_1-\theta_2) = 1+\tan\theta_1\tan\theta_2/\tan\theta_1-\tan\theta_2$

$\Rightarrow \cot^2\beta = (1+\tan\theta_1\tan\theta_2)^2/(\tan\theta_1-\tan\theta_2)^2$

$= (h^2+k^2+b^2-a^2)^2/4(a^2k^2-b^2h^2+a^2b^2)^2$

Simplifying, the required locus is

$(x^2 + y^2 + b^2 - a^2)^2 = 4\cot^2b(a^2y^2 - b^2x^2 + a^2b^2)^2$

Example: Find the equation to the hyperbola whose asymptotes are the straight lines $x + 3y - 1= 0$ and $2x - y + 7 = 0$, and which passes through the point (1, 2).

Solution: Equation to the asymptotes are given as

$x + 3y - 1 = 0$ and …… (1)

$2x - y + 7 = 0$ …… (2)

(1) and (2) may be given by

$(x + 3y - 1)(2x - y + 7) = 0$ …… (3)

As the equation to the hyperbola will differ from (3) only by a constant, it may be given by

$(x + 3y - 1)(2x - y + 7) = \lambda$ …… (4)

(where λ is a constant)

(1, 2) lies on the curve given by (4), we have

$(1 + 6 - 1)(2 - 2 + 7) = \lambda$

$\Rightarrow \lambda = 42$

Hence the equation to the hyperbola will be

$(x + 3y - 1)(2x - y + 7) = 42$

$\Rightarrow 2x^2 - xy + 6xy + 7x - 3y^2 + 21y - 2x + y - 7 = 42$

$\Rightarrow 2x^2 - 3y^2 + 5xy + 5x + 22y - 49 = 0$

Example: Find the equation of the normal at the point (3,4) to the Rectangular Hyperbola $xy = 12$ and the coordinates of its second point of intersection with the curve.

Solution: Here $c^2 = 12 \quad \Rightarrow \quad c = 2\sqrt{3}$

The parameter "t" of the point (3, 4) is given by :-

$3 = 2\sqrt{3}\ t$

$\therefore \quad t = \dfrac{\sqrt{3}}{2}$

The equation of the normal is given by:-

$$\left(\frac{\sqrt{3}}{2}\right)^2 x - y = 2\sqrt{3}\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{2}{\sqrt{3}}\right]$$

Or   3x-4y+7=0

Let the second point of intersection of the Normal be the point $\left(2\sqrt{3}t\ ,\dfrac{2\sqrt{3}}{t}\right)$

and since this point lies on the line 3x - 4y + 7 = 0

$$6\sqrt{3}\ t - \frac{8\sqrt{3}}{t} + 7 = 0$$

This can be written in the form $\left(2t - \sqrt{3}\right)\left(3\sqrt{3}t + 8\right) = 0$

$\therefore \ t = \dfrac{\sqrt{3}}{2}$ or $t = \dfrac{-8}{3\sqrt{3}}$

The first value of t corresponds to the point (3,4) and so the coordinate of the second point of intersection by using second value. With this value for t the point $\left(2\sqrt{3}t\ ,\dfrac{2\sqrt{3}}{t}\right)$ becomes the point $\left(-\dfrac{16}{3}\ ,\ -\dfrac{9}{4}\right)$.