Matrices

A system of any *mn* numbers arranged in a rectangular array of *m* rows and *n* columns is called a matrix.

Order of a matrix

If a matrix contains *m* rows and *n* columns then it is called a matrix of order $m \times n$. Obviously a matrix of order $m \times n$ contains *mn* elements.

Representation of a matrix

A matrix is usually denoted by capital letters *A, B, C*...etc. while its elements are denoted by small letters. *a, b, c*,..etc. The *mn* elements of a matrix are always enclosed by round bracket or by square bracket. For example, $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, B = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}, C = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, D = [2 \;\; 0]$ are the matrices of order $2 \times 3, 2 \times 2, 2 \times 1$ and 1×2 respectively.

A matrix *A* of order $m \times n$ is usually written in the following manner: $A = \begin{bmatrix} a_{11} & a_{12}\dots & a_{1j} & \cdots a_{1n} \\ a_{21} & a_{22}\dots & a_{2j} & \dots a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2}\dots & a_{ij} & \cdots a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2}\dots & a_{mj} & \cdots a_{mn} \end{bmatrix}$

or $A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}], i = 1, \dots, m : j = 1, \dots, n$. From the above representation it is clear that the $(i, j)^{th}$ element of matrix *A* is expressed by a_{ij}.

Various types of matrices

(1) **Row matrix :** If in a matrix, there is only one row, then it is called a row matrix. Obviously, $A = [a_{ij}]_{m \times n}$ is a row matrix, if *m* = 1. For example, [1 3 5] is a row matrix of order 1×3

(2) **Column matrix :** If there is only one column in a matrix, it is called a column matrix. Obviously, $A = [a_{ij}]_{m \times n}$ is a column matrix, if *n* = 1. For example, $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ is a column matrix or order 3×1.

(3) **Singleton matrix :** If a matrix has only one element *i.e.* if for a matrix $A = [a_{ij}]_{m \times n}$ *m* = *n* = 1,then it is called a singleton matrix.

(4) **Null or Zero matrix :** If all the elements of a matrix are zero, then it is called a zero matrix and is denoted by O. Hence $A = [a_{ij}]_{m \times n}$ is a zero matrix, when $a_{ij} = 0, \forall i, j$. For example, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is zero matrix of order 2×3.

(5) **Square matrix :** If number of rows and columns of a matrix are equal, then the matrix is called a square matrix : For example, $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 5 & 7 \\ 0 & 4 & 8 \end{bmatrix}$ is a square matrix. The elements of a square matrix $A = [a_{ij}]_{m \times n}$ for which $i = j$ *i.e.,* $a_{11}, a_{22}, \dots, a_{nn}$ are called the diagonal elements and the line joining the diagonal elements is called the leading diagonal of the matrix. In the above example the diagonal elements are 2, 5, 8.

(6) **Diagonals matrix :** A square matrix is said to be diagonal matrix if all the elements except the principal diagonal elements are zero Hence a square matrix $[a_{ij}]_{n\times n}$ is a diagonal matrix. If $a_{ij} = 0 . i \neq j$

For example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix of order 3×3, which can be denoted by *diag* [2 5 7].

Note : ❑ No element of principal diagonal in diagonal matrix is zero.

(7) **Scalar matrix :** If all the diagonal elements of diagonal matrix are equal, then it is called a scalar matrix, Hence $A = [a_{ij}]_{n\times n}$ is a scalar matrix, if, $a_{ij} = \begin{cases} 0, & i \neq j \\ k, & i = j \end{cases}$. For example, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix.

(8) **Unit matrix :** If all the diagonal elements of a square matrix are equal to one and remaining elements are zero, then it is called a unit matrix, Thus a square matrix $A = [a_{ij}]_{n\times n}$ is a unit matrix. If, $a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$.

(9) **Triangular matrix :** A square matrix $[a_{ij}]$ is said to be a triangular matrix, if each element above or below the leading diagonal is zero. Further a square matrix $[a_{ij}]$ is called the upper triangular matrix, if, $a_{ij} = 0, when\ i > j$. Similarly a square matrix $[a_{ij}]$ is called the lower triangular matrix , if $a_{ij} = 0, when\ i < j$.

Trace of a matrix

The sum of all diagonal elements of a square matrix $A = [a_{ij}]_{n\times n}$ is called the trace of matrix which is denoted by *tr A i.e.* $trA = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{nn}$.

Properties : (1) Trace (*A*) = Trace (A') (2) Trace $(I_n) = n$

(3) Trace (0) = 0 (4) Trace (AB) $\neq$ Trace *A* . Trace *B*

Equal matrices

Two matrices *A* and *B* are said to be equal matrices if and only if they are of same order and their corresponding elements are identical *i.e.* If $A = [a_{ij}]$ and $B = [b_{ij}]$. Then *A* = *B*, if

(1) *order of A =order of B* (2) $a_{ij} = b_{ij}, \forall i\ and\ j$

Addition and subtraction of matrices

(1) **Addition of matrices :** If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are matrices of the same order, then their sum *A* + *B* is a matrix whose each element is the sum of corresponding elements of *A* and *B i.e.*, $A + B = [a_{ij} + b_{ij}]_{m\times n}$

Properties of matrix Addition : If *A*,*B* and *C* are matrices of same order, then

(i) $A + B = B + A$ (Commutative law) (ii) $(A+B)+C = A+(B+C)$ (Associative law)

(iii) $A + 0 = A = 0 + A$ (iv) $A + (-A) = 0 = (-A) + A$, where $(-A)$ is inverse element for matrix addition. 0 is identity elements for matrix addition.

(v) $\left.\begin{array}{l} A+B=A+C \\ B+A=C+A \end{array}\right\} \Rightarrow B=C$ (Cancellation law)

(2) **Subtraction of matrices –** If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are matrices of same order, then their difference *A–B* is a matrix whose each element is the difference of corresponding elements of *A* and *B i.e.,* $A-B=[a_{ij}-b_{ij}]_{m\times n}$

Scalar multiplication of matrix

If *k* is a scalar and *A* is a matrix of order $m\times n$, then *kA is* a matrix of order $m\times n$ whose each element is obtained by multiplying every element of *A* by *k,* i.e. $kA=Ak=[ka_{ij}]_{m\times n}$

(1) **Properties of scalar multiplication :** If *A* and *B* are matrices of the same order and *m*, *n* are any two scalars, then

(i) $m(A+B)=mA+mB$ (ii) $(m+n)A=mA+nA$

(iii) $m(nA)=(mn)A=n(mA)$ (iv) $(-m)A=-(mA)=m(-A)$

Multiplication of matrices

If *A* and *B* be any two matrices, then their product *AB* will be defined only when number of columns in *A* is equal to the number of rows in *B*. If $A=[a_{ij}]$ is a matrix of order $m\times n$ and $B=[b_{ij}]$ is a matrix of order $n\times p$, then their product $AB=C=[c_{ij}]$, will be marix of order $m\times p$, where, $(AB)_{ij}=c_{ij}=(i^{th}\text{row of } A)\times(j^{th}\text{column of } B)$ $=a_{i1}b_{1j}+a_{i2}b_{2j}+....+a_{in}b_{nj}$

$$\therefore c_{ij}=\sum_{r=1}^{n}a_{ir}b_{rj}$$

Properties of matrix multiplication : If *A*, *B* and *C* are three matrices such that their product is defined, then

(i) $AB\neq BA$ (not commutative in general) (ii) $(AB)C=A(BC)$ (Associative law)

(iii) $AI=A=IA$, where *I* is identity matrix for matrix multiplication and *A* is a square matrix.

(iv) $A(B+C)=AB+AC$ (Distributive law)

(v) $AB=AC$, not necessarily implies *B* = *C i.e.* cancellation law is not necessarily true for matrix multiplication.

(vi) *AB* = 0 does not necessarily imply *A* = 0 or *B* = 0 *i.e.* product of two non-zero matrices may be a zero matrix.

Remember that if *A* and *B* are two matrices of the same order, then

(a) $(A+B)^2=A^2+B^2+AB+BA$ (b) $(A-B)^2=A^2+B^2-AB-BA$

(c) $(A-B)(A+B)=A^2-B^2+AB-BA$ (d) $(A+B)(A-B)=A^2-B^2-AB+BA$ (e) $A(-B)=(-A)B=-(AB)$.

Positive integral power of matrix

If *A* is any matrix, then the product *A.A* is defined only when *A* is a square matrix. *A*. *A* is written as A^2. Similarly we write $A^2.A$ as A^3. In general we have, $A.A.A.....A(n\text{ factors})=A^n$

For arbitrary positive integers *m, n* we have

(1) $A^m A^n = A^{m+n}$ (2) $(A^m)^n = A^{mn} = (A^n)^m$

(3) $I^m = I$ (4) $A^0 = I_n$, where A is a square matrix of order n.

Transpose of a matrix

The matrix obtained from a given matrix A by a changing its rows into columns and columns into rows is called the transpose of A and is denoted by A^T or A' If A is a matrix of order $m\times n$ then A^T or A' will be a matrix of order $n\times m$.

Properties of transpose : If A and B are matrices of suitable orders then

(1) $(A\pm B)^T = A^T \pm B^T$

(2) $(AB)^T = B^T A^T$

In general, if $A_1, A_2, ..., A_n$ are matrices of suitable order, then $(A_1 A_2 .... A_n)^T = A_n^T A_{n-1}^T .... A_2^T A_1^T$

(3) $(A^T)^T = A$ (4) $(KA)^T = K.A^T$, where K is a scalar (5) $I^T = I$

Special matrix

(1) **Symmetric matrix :** A square matrix $A=[a_{ij}]$ is called symmetric matrix, if $a_{ij}=a_{ji}, \forall i,j$ or $A^T = A$

(2) **Skew-symmetric matrix :** A square matrix $A=[a_{ij}]$ is called a skew-symmetric matrix, if $a_{ij}=-a_{ji}, \forall\, i,j$ or $A^T=-A$.

(3) **Hermitian matrix :** Square matrix $A=[a_{ij}]$ is said to be hermitian matrix if $A=A^\theta$.

(4) **Skew hermitian matrix :** Square matrix $A=[a_{ij}]$ is said to be skew hermitian matrix, if $A^\theta = -A$.

Adjoint matrix

If F_{ij} is the cofactor of element a_{ij} in the determinant $|A|$ of a square matrix A, then the transpose of matrix $[F_{ij}]$ is called the adjoint matrix of A and is denoted by (*adj* A). Obviously (i, j^{th}) element of *adj* A=cofactor of a_{ji} in $|A|$.

Thus, if $A=\begin{bmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ ... & .... & ... \\ a_{n1} & a_{n2} & ...a_{nn}\end{bmatrix}$, then $adj\ A=\begin{bmatrix} F_{11} & F_{12} & ...F_{1n} \\ F_{21} & F_{22} & ...F_{2n} \\ ... & .... & ... \\ F_{n1} & F_{n2} & ...F_{nn}\end{bmatrix}^T, =\begin{bmatrix} F_{11} & F_{21} & ...F_{1n} \\ F_{12} & F_{22} & ...F_{n2} \\ ... & .... & ... \\ F_{1n} & F_{2n} & ...F_{nn}\end{bmatrix}$.

Properties of adjoint matrix

If A and B are square matrices of order n. Then, (i) $A.(adj\ A) = |A|\ I_n = (adj\ A).A$

(ii) $|adj\ A| = |A|^{n-1}$ (iii) $adj.adjA = |A|^{n-2} A\ [\because |A| \neq 0]$

(iv) $|adj.adj(A)| = |A|^{(n-1)^2}$ (v) $(adj\ A)^T = adj\ A^T$

(vi) $adj(AB) = (adj\ B).(adj\ A)$ (vii) $adj(A^n) = (adj\ A)^n, n \in N$

(viii) $adj(kA) = k^{n-1}(adj\ A), k \in R$ (ix) $adj(0) = 0$

(x) $adj(I_n) = I_n$

(xi) A is symmetric matrix $\Rightarrow$ *adj* A is a symmetric matrix.

(xii) A is diagonal matrix $\Rightarrow$ *adj* A is a diagonal matrix

(xiii) A is triangular matrix $\Rightarrow$ *adj* A is a triangular matrix.

Conjugate of a matrix

When each element of a matrix is replaced by their conjugate complex number, then obtained matrix is called conjugate matrix. It is denoted by $\overline{A}$. If $A = [a_{ij}]$, then $\overline{A} = [\overline{a}_{ij}]_{m \times n}$.

Properties of conjugate matrix

(1) $(\overline{\overline{A}}) = A$ (2) $(\overline{AB}) = \overline{A}\,\overline{B}$ (3) $(\overline{\lambda A} = \overline{\lambda}\,\overline{A})$ (4) $(\overline{A \pm B}) = (\overline{A} \pm \overline{B})$

Conjugate transpose matrix

Transpose of conjugate matrix of any matrix is called conjugate transpose matrix. It is denoted by $(\overline{A})^T = A^{\theta} = A^{*}$.

Properties

(1) $(A^{\theta})^{\theta} = A$ (2) $(A \pm B)^{\theta} = A^{\theta} \pm B^{\theta}$ (3) $(AB)^{\theta} = B^{\theta} A^{\theta}$ (4) $(\lambda A)^{\theta} = \overline{\lambda}\, A^{\theta}$

Inverse matrix

If A is a square matrix of order n and a matrix B exists so that, $AB = I = BA$ .Where I is a unit matrix of order n, then matrix B is called the inverse matrix of A and is denoted by A^{-1}. To find inverse matrix of a given matrix A, we use the following formula: $A^{-1} = \frac{adj\,A}{|A|}$, $[\because |A| \neq 0]$.

Here we see that inverse of a matrix exists only when $|A| \neq 0$ and inverse matrix is always unique.

The square matrix A for which $|A| \neq 0$ is called non-singular matrix and the matrix A for which $|A| = 0$ is called a singular matrix.

Properties of inverse matrix

Let A and B are two invertible matrices of the same order, then

(1) $(A^T)^{-1} = (A^{-1})^T$

(2) $(AB)^{-1} = B^{-1}A^{-1}$

(3) $(A^k)^{-1} = (A^{-1})^k, k \in N$

(4) $adj(A^{-1}) = (adj\,A)^{-1}$

(5) $(A^{-1})^{-1} = A$

(6) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

(7) If $A = diag(a_1, a_2, \ldots\ldots\ldots a_n)$, then $A^{-1} = diag(a_1^{-1}, a_2^{-1}, \ldots. a_n^{-1})$

(8) A is a symmetric matrix $\Rightarrow A^{-1}$ is a symmetric matrix.

(9) A is a triangular matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is a triangular matrix.

(10) A is a scalar matrix $\Rightarrow A^{-1}$ is a scalar matrix.

(11) A is a diagonal matrix $\Rightarrow A^{-1}$ is a diagonal matrix.

(12) $AB = AC \Rightarrow B = C$, iff $|A| \neq 0$.

Some special kinds of matrices

(1) **Orthogonal matrix :** If for any square matrix *A*, $A^{-1} = A^T$ *i.e.*, $AA^T = I$, then *A* is called **orthogonal matrix.**

(2) **Idempotent matrix :** If for a square matrix *A*, $A^2 = A$. then *A* is called the **idempotent matrix.**

(3) **Involutory matrix :** If for a square matrix *A*. $A^2 = I$ or $A^{-1} = A$. Then *A* is called the **involutory matrix.** Every unit matrix is a involutory matrix.

(4) **Nilpotent matrix :** A square matrix *A* is called a Nilpotent matrix if there exists $p \in N$ such that $A^p = 0$.

(5) **Singular matrix :** A square matrix $A = [a_{ij}]$ is a singular matrix, if $|A| = 0$.

(6) **Non-singular matrix :** A square matrix $A = [a_{ij}]$ is a non-singular if $|A| \neq 0$.

Rank of matrix

A number *r* is said to be rank of matrix *A* if

(1) Every square submatrix of order (*r*+1) or more is singular,and

(2) There exists atleast one square submatrix of order *r* which is non-singular. Thus, the rank of a matrix is the order of the highest order non-singular square submatrix.

System of simultaneous linear equations

Consider the following system of *m* linear equations in *n* unknowns

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ &\dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \\ &a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{aligned} \qquad ....(i)$$

This system of equations can be written in matrix form as $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ or $AX = B$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$ The $m \times n$ matrix *A* is called the coefficient matrix of the system of linear equations. A set of values of the variable $x_1, x_2, ... x_n$ which simultaneously satisfy all the equations is called a solution of the system of equation,.

(1) **Consistent system :** If the system of equations has one or more solutions, then it is said to be consistent system of equations otherwise it is an inconsisent system of equations.

(2) **Homogeneous and Non-Homogeneous systems of linear equations :** A system of equations $AX = B$ is called a homogeneous if $B = 0$. Otherwise,it is called a non-homogeneous system of equations.

(3) **Solution of a non-homogenous system of linear equations :** There are three method of solving a non-homogenous system of simultaneous linear equations.

(i) Determinant method (Cramer's rule) (ii) Matrix method (iii) Rank method

Matrix method

Let $AX = B$ be a system of n linear equations with n unknowns. If A is non-singular, then A^{-1} exists.

$\therefore AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$ [pre-multiplying by A^{-1}]

$\Rightarrow (A^{-1}A)X = A^{-1}B$ [by associativity] $\Rightarrow I_n X = A^{-1}B \Rightarrow X = A^{-1}B$

Thus, the system of equations $AX = B$ has a solution given by $X = A^{-1}B$. Thus, if A is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$.

Algorithm for solving a non-homogeneous system of linear equations

Let $AX = B$ be a non-homogeneous system of 3 linear equations in 3 unknowns.To solve this system of equations we (proceed as follows)

Step 1 : Write the given system of equations in matrix form $AX = B$ and obtain A, B.

Step 2 : Find $|A|$

Step 3 : If $|A| \neq 0$, then write "the system is consistent with unique solution". Obtain the unique solution by the following procedure. Find A^{-1} by using $A^{-1} = \frac{1}{|A|} adj\, A$. Obtain the unique solution given by $X = A^{-1}B$.

Step 4 : If $|A| = 0$ then write "the system is either consistent with infinitely many solutions or it is inconsistent. To distinguish these two proceed as follow : Find $(adj\, A)\, B$

If $(adj\, A)B \neq 0$ then write "the system is inconsistent". If $(adj\, A)B = 0$, then the system is consistent with infinitely many solutions. To find these solutions proceed as follows. Put $z = k$ (any real number) and take any two equations out of three equations. Solve these equations for x and y. Let the values of x and y be λ and μ respectively. Then, $x = \lambda,\ y = \mu,\ z = k$ is the required solution.

Cayley-Hamilton theorem

Every matrix satisfies its characterstics equation. *e.g.* let A be a square matrix then $|A - xI| = 0$ is the characterstics equation of A. If $x^3 - 4x^2 + 5x - 7 = 0$ is the characterstics equation for A, then $A^3 - 4A^2 + 5A - 7I = 0$.

Roots of characterstics equation for A are called eigen values of A or characterstics roots of A or latent roots of A.

If λ is characterstics root of A, then λ^{-1} is characterstics root of A^{-1}.

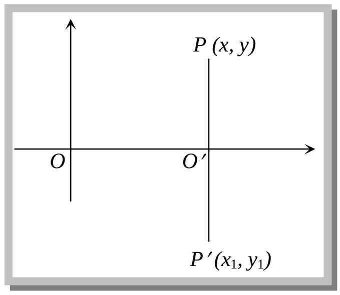
Geometrical transformations

(1) **The Reflection matrix :**

(i) **Reflection in the *x*-axis** : Let $P(x,y)$ be any point and $P'(x_1,y_1)$ be its image after reflection in the *x*-axis, then $\begin{cases} x_1 = x \\ y_1 = -y \end{cases}$ (O' is the midpoint of *P* and P' .

$\begin{cases} x_1 = 1.x + 0.y \\ y_1 = 0.x + (-1)y \end{cases}$. These system of equations in the matrix form is, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Thus the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point $P(x,y)$ in the *x*-axis.



(ii) Reflection in the *y*-axis : Here the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(iii) Reflection through the origin : Here the matrix is $\begin{bmatrix} -1 & 0 \\ 0- & 1 \end{bmatrix}$.

(iv) Reflection in the line $y = x$: Here the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(v) Reflection in the line $y = x \tan\theta$: Here the matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(2) **Rotation through an angle θ :** Here the matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.