# MOTION IN TWO AND THREE DIMENSION

## Contents

# 2. MOTION IN TWO AND THREE DIMENSION

## 2.2 SCALARS AND VECTORS

### 2.2.1 Scalars

Physical quantities which require only the magnitude (numerical value) for their complete representation are called scalar quantities. They don't have any direction. Mass, Energy, Temperature are some examples of scalar quantities.

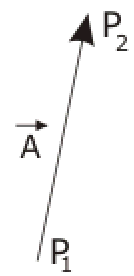
### 2.2.2 Vectors

Physical quantities which require magnitude and direction both are called vector quantities. Force, Momentum, Electric field etc. are vector quantities. Vectors are usually represented by a line segment with an arrow.

#### 2.2.2.1 Representation of a Vector

Since vectors have directions, any representation of them has to include the direction.

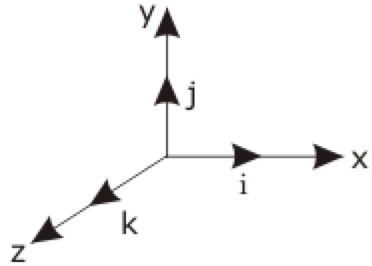
To represent a vector we use a line with an arrow head. The length of the line represents the magnitude of vector and direction of the arrow represents the direction of the vector.




Vector is a Physical quantity and all physical quantities have units.

#### 2.2.2.2 Unit Vectors

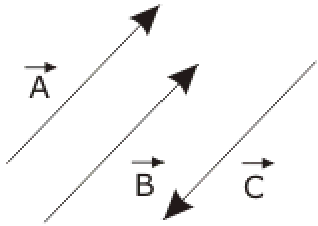
A unit vector is a vector having a **magnitude of unity.** Its only purpose is to describe a direction in space. On x-y co-ordinate system $\hat{\imath}$ denote unit vector in positive x direction and $\hat{\jmath}$ denotes unit vector in positive y direction.



Any vector in x – y plane can be represented in terms of these unit vectors $\hat{\imath}$ & $\hat{\jmath}$.

Similarly any vector in a 3 dimensional x y z space can be represented in terms of unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ where, $\hat{k}$ is the unit vector in the positive z direction, as shown in figure above.

**2.2.2.3 Parallel Vectors:** Two or more vectors are said to be parallel when they are parallel to the same line. In the figure below, the vectors A B and C are all parallel.




**2.2.2.4 Equal Vectors:** Two or more, vectors are equal if they have the same magnitude (length) and direction, whatever their initial points. In the figure above, the vectors A and B are equal.

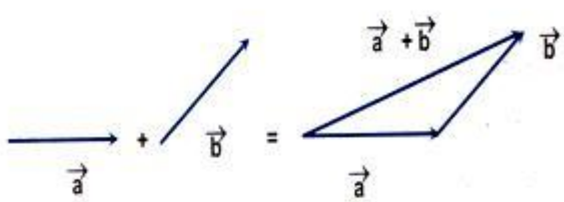
**2.2.2.5 Negative Vectors:** Two vectors which have same magnitude (length) but their direction is opposite to each, other called the negative vectors of each other. In figure above vectors A and C or B and C are negative vectors.

**2.2.2.6 Null Vectors:** A vector having zero magnitude is called zero vector or 'null vector' and is written as 0 vector. The initial point and the end point of such a vector coincide so that its direction is indeterminate.

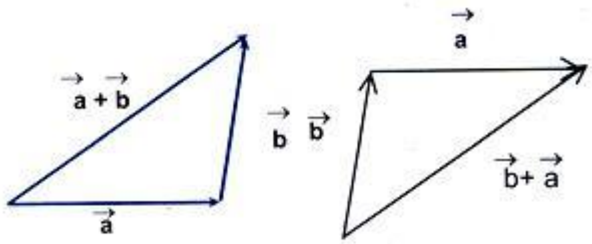
**2.2.2.7 Invariance of the vector:** Any vector is invariant so it can be taken anywhere in the space keeping its magnitude and direction same**.** In other words, the vectors remain invariant under translation.

**2.2.3 Addition and subtraction of vectors**

**2.2.3.1 Geometrical method**

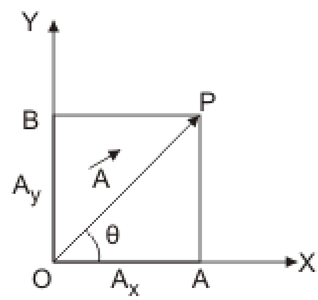



To find $\vec{a} + \vec{b}$, shift vector $\vec{b}$ such that its initial point coincides with the terminal point of vector $\vec{a}$. Now, the vector whose initial point coincides with the initial point of vector $\vec{a}$, and terminal point coincides with the terminal point of vector $\vec{b}$ represents $\vec{a} + \vec{b}$ as shown in the figure.




#### 2.2.3.2 Resolution of a vector

Consider a vector $\vec{A}$ in X – Y Plane represented by OP which is inclined at an angle $\theta$ with X-axis. Its projection along OX and OY are OA and OB respectively. The projections are known as X and Y components of the vector $\vec{A}$.




Thus

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

where $A_x = A\cos\theta \; and \; A_y = A\sin\theta$

Further, we observe $A = \sqrt{A_x^2 + A_y^2}$

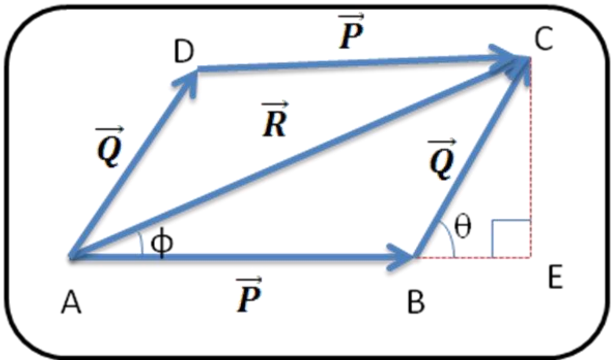
and $\tan\theta = \frac{A_y}{A_x}$

If the vector $\vec{A}$ be in space which has three components then

$$\vec{A} = \pm A_x\hat{\imath} \pm A_y\hat{\jmath} \pm A_z\hat{k} \text{ and } A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

#### 2.2.3.3 Law of Parallelogram of Vectors

According to this law if two vectors $\vec{P}$ and $\vec{Q}$ are represented by two adjacent sides of a parallelogram both pointing outwards as shown in the figure below , then the diagonal drawn through the intersection of the two vectors represents the resultant (i.e. vector sum of $\vec{P}$ and $\vec{Q}$).




In case of addition of two vectors by parallelogram method as shown in figure, the magnitude of resultant will be given by,

$$(AC)^2 = (AE)^2 + (EC)^2$$

$$\text{or } R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$\text{or } R = \sqrt{(P^2 + Q^2) + 2PQ\cos\theta}$$

And the direction of resultant from vector P will be given by

$$\tan\Phi = CE/AE = Q\sin\theta / (P+Q\cos\theta)$$

$$\Phi = \tan^{-1}[Q\sin\theta / (P+Q\cos\theta)]$$

### 2.2.3.4 Vector subtraction

Suppose there are two vectors A and B, shown in figure A and we have to subtract B and A. It is just the same thing as adding vectors – B to A. The resultant is shown in figure B.

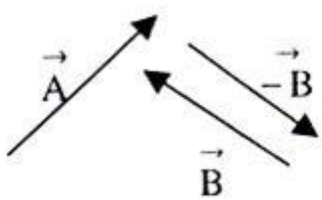



**Figure (A)**

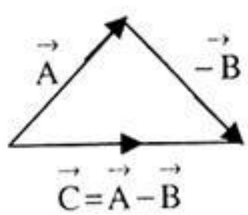



**Figure (B)**

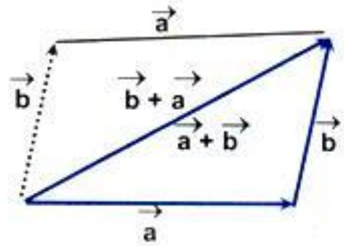
$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

### 2.2.3.5 Properties of Vector Addition

**1. Vector addition is commutative**

i.e

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$




**2. Vector addition is associative**

i.e

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$

| NOTE | |
|---|---|
| 1. | Only like vectors are added together, for example, a displacement vector can be added to another displacement vector. A velocity vector can be added to another velocity vector. A displacement vector cannot be added to a velocity vector. |
| 2. | In simple geometry, we know that sum of the lengths of two sides of a triangle is greater than the length of the third side. In a vector triangle, the sum of the magnitudes of two vectors is greater than the magnitude of their resultant, provided they are not in the same direction. |
| 3. | When a positive number is multiplied by a unit vector it becomes a vector of magnitude equal to the magnitude of the number in the direction of the given unit vector. When a unit vector is multiplied by a negative number the result is a vector whose magnitude equals magnitude of the number but whose direction is opposite to that of given unit vector. |

### 2.2.4 Multiplication of Vectors

### 2.2.4.1 Multiplication of vector by a scalar

Let vector $\vec{a}$ is multiplied by a scalar m. If m is a positive quantity, only magnitude of the vector will change by a factor 'm' and its direction will remain same. If m is a negative quantity the direction of the vector will be reversed.

### 2.2.4.2 Multiplication of a vector by a vector

(i) Dot product or scalar product
(ii) Cross product or vector product

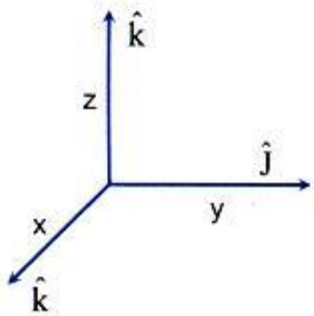
### 2.2.4.3 Dot product or scalar product

The dot product of two vectors a and b is defined as

$$\vec{a} \cdot \vec{b} = ab\cos\theta$$

where a and b are the magnitudes of the respective vectors and θ is the angle between them. The final product is a scalar quantity. If two vectors are mutually perpendicular then $\theta = 90^0$ and $\cos 90^0 = 0$, Hence, their dot product is zero.

Some examples of dot product: work = $\vec{F} \cdot \vec{s} = Fs\cos\theta$




Here,

$$\hat{\imath}.\hat{\jmath} = 0,\ \hat{\imath}.\hat{k} = 0, \&\ \hat{\jmath}.\hat{k} = 0 \text{ and } \hat{\imath}.\hat{\imath} = 1,\ \hat{\imath}.\hat{\jmath} = 1,\ \&\ \hat{k}.\hat{k} = 1$$

**Geometrical meaning of dot product**

As $\vec{a}.\vec{b} = ab\cos\theta = a(b\cos\theta) = (a\cos\theta)\,b$

The dot product of $\vec{a}$ and $\vec{b}$ is equal to the product of $a$ and projection of $\vec{b}$ on $\vec{a}$ or product of $b$ and projection of $\vec{a}$ and $\vec{b}$.

**Properties of scalar product or dot product**

(i) Dot product is commutative, $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

(ii) Dot product is distributive over vector addition.

$$\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$$

***Example***: A particle acted upon by constant forces $2\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$ and $-\hat{\imath} - 2\hat{\jmath} - \hat{k}$ is displaced from the point $4\hat{\imath} - 3\hat{\jmath} - 2\hat{k}$ to the point $6\hat{\imath} + \hat{\jmath} - 3\hat{k}$.. Find the total work done by the forces.

***Solution***: Let $\vec{F}$ be the resultant of the given forces and $\vec{s}$ be the displacement.

Then, $\vec{F} = (2\hat{\imath} + 5\hat{\jmath} + 6\hat{k}) + (-\hat{\imath} - 2\hat{\jmath} - \hat{k}) = \hat{\imath} + 3\hat{\jmath} + 5\hat{k}$

And, $\vec{s} = (6\hat{\imath} + \hat{\jmath} - 3\hat{k}) - (4\hat{\imath} - 3\hat{\jmath} - 2\hat{k}) = 2\hat{\imath} + 4\hat{\jmath} - \hat{k}$

The total work done

$\vec{F}.\vec{s} = (\hat{\imath} + 3\hat{\jmath} + 5\hat{k}).(2\hat{\imath} + 4\hat{\jmath} - \hat{k}) = 9$ units.

***Example***: Find the projection of $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}$ along the vector $\vec{B} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$.

***Solution***: The projection of vector $\vec{A}$ in the direction of vector $\vec{B}$ is given by

P=$|\vec{A}|\cos\theta$ ...(i)

where $\theta$ = angle between the $\vec{A}$ and $\vec{B}$

We know that, $\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$$|\vec{A}|\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{B}|}$$

Equation (i) becomes

$$P = \frac{\vec{A}.\vec{B}}{|\vec{B}|}$$

$$P = \frac{(\hat{\imath} + \hat{\jmath} + \hat{k}).(2\hat{\imath} - \hat{\jmath} + \hat{k})}{\sqrt{2^2 + 1^2 + 1^2}}$$

$$p = \frac{(2 - 1 + 1)}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

***Example***: If $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$. Find (a) The scalar product $\vec{a}.\vec{b}$ and (b) The angle between them

***Solution:*** $\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z$

= 2 + 8 + 3 = 13

If the angle between $\vec{a}$ and $\vec{b}$ be $\theta$

Then $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$$= \frac{13}{\sqrt{4 + 16 + 9}\,\sqrt{1 + 4 + 1}} = \frac{13}{\sqrt{29 \times 6}} = \frac{13}{\sqrt{174}}$$

***Example:*** If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ show that $\vec{a} \perp \vec{b}$.

***Solution***: We have $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}).(\vec{a} + \vec{b})$

$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}$

$= a^2 + b^2 + 2\vec{a}\vec{b}$

Similarly,

$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}).(\vec{a} - \vec{b})$

$= a^2 + b^2 - 2\vec{a}\vec{b}$

If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

$a^2 + b^2 + 2\vec{a}\vec{b} = a^2 + b^2 - 2\vec{a}\vec{b}$

or, $\vec{a}.\vec{b} = 0$

or, $\vec{a} \perp \vec{b}$

### 2.2.4.4 Vector product (cross product)

The vector product of two vectors $\vec{a}$ and $\vec{b}$ inclined at angle $\theta$ is a vector quantity whose magnitude is given by $ab\sin\theta$ and its direction perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. It is also known as the cross product and is expressed as $\vec{a}\times\vec{b} = ab\sin\theta\ \hat{n}$ where $\hat{n}$, is the unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ (according to the right handed screw rule). Here the angle $\theta$ is such that $0° < \theta < 180°$.

**Vector product of orthogonal unit vectors** $(\hat{\imath}, \hat{\jmath}, \hat{k})$

In the right handed coordinate system, coordinate axes $x, y$ and $z$ are so chosen that bending the fingers of the right hand from X to Y will lead the thumb along the $z$ -axis. For such a system

$\hat{\imath}\times\hat{\jmath} = \hat{k}$ $\qquad$ $\hat{\jmath}\times\hat{\imath} = -\hat{k}$

$\hat{k}\times\hat{\imath} = \hat{\jmath}$ $\qquad$ $\hat{\imath}\times\hat{k} = -\hat{\jmath}$

$\hat{\jmath}\times\hat{k} = \hat{\imath}$ $\qquad$ $\hat{k}\times\hat{\jmath} = -\hat{\imath}$

$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = \vec{0} =$ zero vector.

Cross product of two vectors in terms of the components along the coordinate axes:

Let $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$

$\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$

then $\quad \vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\hat{\imath} + (a_z b_x - a_x b_z)\hat{\jmath} + (a_x b_y - a_y b_x)\hat{k}$

This can be represented in determinant form as $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

***Example***: If $\vec{A} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and
$\vec{B} = \hat{\imath} + \hat{\jmath} + \hat{k}$, Find $\vec{A}\times\vec{B}$
What are the $x,\ y$ and $z$ components of $(\vec{A}\times\vec{B})$?

***Solution***: Expressing the vector product in determinant form,

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 5 \\ 1 & 1 & 1 \end{vmatrix}$$

$= (4-5)\hat{\imath} + (5-3)\hat{\jmath} + (3-4)\hat{k}$

$= -\hat{\imath} + 2\hat{\jmath} - \hat{k}$

The $x,\ y$ and $z$ components of $\vec{A}\times\vec{B}$ are -1, +2 and -1 respectively.

**Properties of vector product or cross product**

1. Cross product of two vectors does not obey the commutative law:
   $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

2. Cross product of two vectors is distributive over vector addition
   $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

**Some of the examples of physical quantities given by cross product of two vectors**

(i) Velocity

$\vec{v} = \vec{\omega} \times \vec{r}$

Where $\vec{\omega}$ = angular velocity, $\vec{r}$ = radius vector

(ii) Angular momentum

$\vec{L} = \vec{r} \times \vec{p}$

Where $\vec{r}$ = positive vector, $\vec{p}$ = linear momentum

(iii) Torque

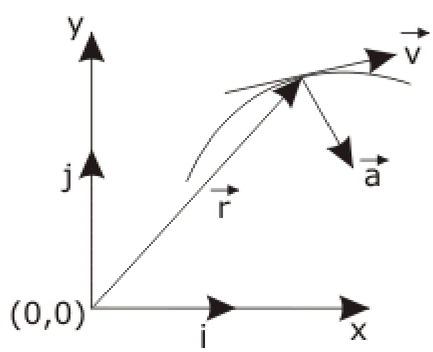
$\vec{\tau} = \vec{r} \times \vec{F}$

where $\vec{F}$ = force

## 2.3. MOTION IN TWO DIMENSIONS

In this part, we discuss motions in two dimensions like the motion of a particle moving on a circular path or on a parabolic path.

Consider a particle moving on x-y plane along a curved path at time t, as shown in figure given below. Its displacement from origin is measured by vector $\vec{r}$, its velocity is indicated by vector $\vec{v}$ (tangent to the path of the particle) and acceleration $\vec{a}$ as shown in the figure.




The vectors $\vec{r}$, $\vec{v}$ and $\vec{a}$ are inter related and can be expressed in terms of their components, using unit vector notation as,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}, \qquad v = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = v_x\hat{\imath} + v_y\hat{\jmath} \qquad and\ \ \vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = a_x\hat{\imath} + a_y\hat{\jmath}$$

From the above relations we can say that two-dimensional motion is nothing but superimpose of two one dimension motions. This is for the reason that in any two-dimensional motion, all the parameters can be resolved in two mutually perpendicular directions.

For a particle moving along a straight line, all the vector quantities position, velocity, displacement and acceleration have only one non-zero component and hence can be treated as positive or negative numbers.
When a particle is moving along a curve, each of these quantities can have two non-zero components. Hence the motion is said to be two dimensional because two numbers (components) are associated with a vector quantity. We will be studying two types of curvilinear motions:

(i) Projectile Motion : motion of a particle under the effect of earth's gravity
(ii) Circular Motion : motion of a particle along a circle

***Example***: A particle starts with initial velocity $(3\hat{i} + 4\hat{j})$ m/s and with constant acceleration $(8\hat{i} + 6\hat{j})$ m/s$^2$. Find the final velocity and final displacement of the particle after time t = 4 sec.

***Solution:*** As the velocity and acceleration vectors are not parallel, the velocity and acceleration cannot be treated as a scalar.
Now divide the whole phenomenon in two one-dimension motions.

**In x-direction**
Initial velocity = $v_{x0}$ = 3m/s, $a_x$ = 8 m/s$^2$
Hence velocity after 4 second, $v_x = 3 + 8 \times 4 = 35$ m/s
Displacement after 4 second
$s_x = x = 3 \times 4 + \frac{1}{2} 8\, (4)^2 = 76$ m

**In y-direction**
Initial velocity $v_{y0}$ = 4m/s, $a_y$ = 6 m/s$^2$
Velocity after 4 second $v_y = 4 + 6 \times 4 = 28$ m/s
Displacement after 4 second $s_y = 4 \times 4 + \frac{1}{2}\, 6\, (4)^2 = 64$ m
It means final velocity $v = v_x\, i + v_y\, j = (35\, i + 28\, j)$ m/s
and displacement = $s = s_x\, i + s_y\, j = (76\, i + 64\, j)$ m

### 2.3.1 Projectile

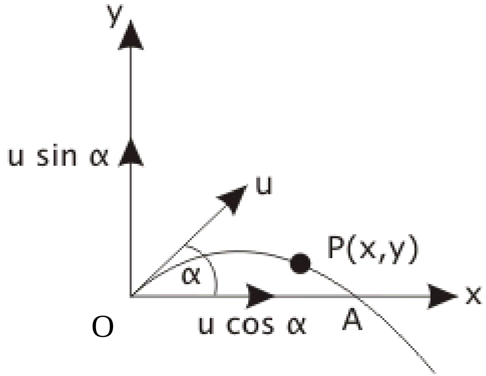
A **projectile** is any object propelled through space by the exertion of a force which ceases after launch.

### 2.3.2 Motion of Projectile

To analyze the projectile motion we use the following concept "Resolution of two dimensional motion into two one dimension motion". Hence it is easier to analyze the motion of projectile as composed of two simultaneous rectilinear motions which are independent of each other :

(a) Along the vertical y-axis with a uniform downward acceleration 'g' and
(b) Along the horizontal x-axis with a uniform velocity forward.

Consider a particle projected with an initial velocity u at an angle α with the horizontal x-axis as shown in figure shown below. Velocity and accelerations can be resolved into two components:




Velocity along x-axis = $u_x$ = u cosα
Acceleration along x-axis $a_x$ = 0
Velocity along y-axis = $u_y$ = u sinα
Acceleration along y-axis $a_y$ = -g

Here we use different equation of motions of one dimension derived earlier to get the different parameters.

$$\vec{v} = \vec{v}_0 - \vec{g}t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

$$\vec{y} - \vec{y}_0 = \vec{v}_0 t - \frac{1}{2}\vec{g}t^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (B)$$

$$v^2 = {v_0}^2 - 2g(y - y_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (C)$$

**Total Time of flight:** When body returns to the same horizontal level, the resultant displacement in vertical y-direction is zero. Use equation (B).

Therefore, $0 = (u \sin \alpha)\, t - (½)gt^2$

or $t = 2u \sin\alpha /g$ (as t cannot equal to 0)

**Horizontal Range:** Horizontal Range (OA) = Horizontal velocity × Time of flight

= $u \cos \alpha \times 2 u \sin\alpha /g$

= $u^2 \sin 2\alpha/g$

**Maximum Height:** At the highest point of the trajectory, vertical component of velocity is zero.

Therefore, $0 = (u \sin\alpha)^2 - 2g H_{max}$

Or, $H_{max} = u^2 \sin^2\alpha/2g$

**Equation of Trajectory:** Assuming the point of projection as the origin of co-ordinates and horizontal direction as the x-axis and vertical direction as the y-axis. Let P (x, y) be the position of the particle at instant after t second.

Then $x = u \cos\alpha . t$ and $y = u \sin\alpha . t - 1/2\, gt^2$

Eliminating 't' from the above equations, we get,

$y = x \tan\alpha - gx^2/2u^2\cos^2\alpha$

This is the equation of trajectory which is a parabola ($y = ax + bx^2$).

***Example:*** A body is projected upwards with a velocity 98 m/s. Find (a) the maximum height reached, (b) the time taken to reach the maximum height, (c) its velocity at a height 196 m from the point of projection, (d) velocity with which it will cross down the point of projection and (e) the time taken to reach back the point of projection.

***Solution:***

a) Initial upward velocity u = 98 m/s

Acceleration a = (-g) = - 9.8 $m/s^2$

Maximum height reached H is given by

$$v^2 = u^2 + 2aS$$

$$0 = 9.8^2 + 2(-9.8)\,H$$

$$H = \frac{98^2}{2\times 9.8} = 490\ m$$

b) $\because \quad v = u + at$

at maximum height $v = 0$

$\Rightarrow \quad 0 = 98 - 9.8t$

$$t = \frac{u}{g} = \frac{98}{9.8} = 10s$$

c) Velocity at a height of 196 m from the point of projection

$$v^2 = u^2 + 2aS$$

$$v^2 = (98)2 + 2(-98)196$$

$$v = \pm\sqrt{5762.4} = \pm 75.91\ m/s$$

d) Velocity with which it will cross down the point of projection

$$v^2 = u^2 + 2gS$$

At the point of projection S = 0

$$v = \pm u$$

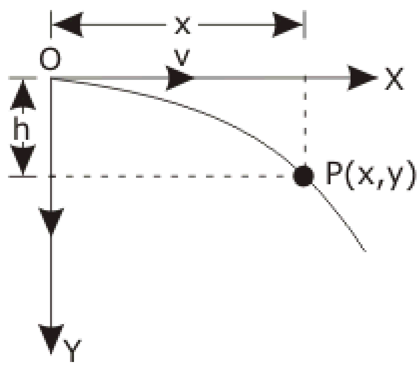
While crossing the point of projection downwards, $v = -u = -98$ m/s. The velocity has the same magnitude as the initial velocity but reversed in direction.

e) The time taken to reach back the point of projection

$$t = \frac{2u}{g} = \frac{2\times 98}{9.8} = 20s$$

### 2.3.2.1 Horizontal projection

Consider a particle projected horizontally with a velocity $\vec{u}$ from a point O as shown in figure given below.




Assuming the point of projection O as the origin of coordinates and horizontal direction as the X-axis and vertical direction as Y-axis. Let P (x, y) be the position of the particle after t seconds.

∴ x = horizontal distance covered in time t = ut. .............. (1)

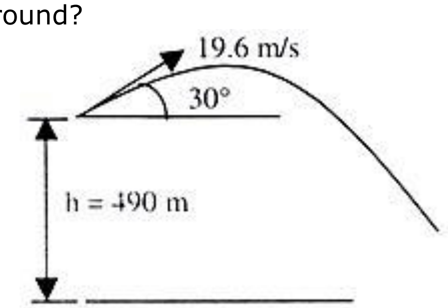
y = vertical distance covered in time t = $\frac{1}{2}gt^2$ .............. (2)

Eliminate t from equations (1) and (2) then

We get, y = $(1/2)(gu^2)/(x^2)$

This is the equation of parabola passing through the origin, with its vertex at the origin O. Hence the trajectory is a parabola.

***Example:*** A stone is thrown at a speed of 19.6 m/sec at an angle 30° above the horizontal from a tower of height 490 meter. Find the time during which the stone will be in air. Also find the distance from the foot of the tower to the point where stone hits the ground?




***Solution*:** Let us consider the motion of stone in the horizontal and vertical directions separately.

(i) Vertical motion (downward direction negative):

Initial vertical velocity y = 19.6 sin 30°

Acceleration a = g = -9.8 m/s$^2$

Vertical distance covered = h = 490 m

Using, $h = ut + 1/2gt^2$

We have, $490 = -9.8t + (1/2)\, 9.8t^2$

$100 = -2t + t^2$ or $t^2 - 2t - 100 = 0$

t = $= \frac{2 \pm \sqrt{(2^2 - 4 \times 1 \times [-100])}}{2 \times 1} = 1 + \sqrt{101}$

∴ t = 11.25 sec

(ii) Horizontal motion:

Initial horizontal velocity y = 19.6 sin 30$^o$ = 9.8 m/s

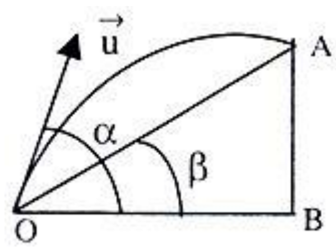
Hence distance from the foot of tower to the point where stone hits the ground

= Horizontal component × time of flight

= 19.6 cos 30 × 11.02 = 190 m

**2.3.2.2 Projectile Motion on an inclined plane**

Let the particle strike the plane at A so that OA is the range of the projectile on inclined plane. This initial velocity $\vec{u}$ can be resolved into two components:




(i) u cos (α - β) along the plane
(ii) u sin (α - β) perpendicular to the plane.

The acceleration due to gravity g can be resolved into two components:

(i) g sin β parallel to the plane
(ii) g cos β perpendicular to the plane.

**Time of Flight:** Let t be the time taken by the particle to go from A to B. In this time the displacement of the projectile to the plane is zero.

Hence, $0 = u \sin(\alpha - \beta)\, t - ½ g\, \beta\, t^2$

=> $t = 2u \sin(\alpha-\beta)/g\cos\beta$

**Range:** During time of flight, the horizontal velocity u cos α remains constant.

Hence, Horizontal distance

$OB = (u\cos\alpha)\, t = 2u^2\sin(\alpha - \beta)\cos\alpha / g\cos\beta$

Now, $OA = OB/\cos\beta = 2u^2\sin(\alpha - \beta)\cos\alpha / g\cos\beta$

**Note:** The greatest distance of the projectile from the inclined plane is $u^2\sin^2(\alpha - \beta)/2g\cos\beta$ .

***Example***: From a point O on an inclined plane of inclination $'\alpha'$ a ball is projected with speed $V_0$ at an angle of $\alpha$ with the normal to the plane down the inclined plane. At what distance from the point of projectile ball will hit the plane.

***Solution***: This is the case of down the plane projection with speed $V_0$ and angle $\alpha$ normal to the plane.

Let us break the motion along two mutually perpendicular components. One along the incline (say $x$) and one along normal (say $y$)

Along $x$ -axis $u_x = v_0 \sin\alpha$ , $a_x = g\sin\alpha$

Along $y$-axis $u_y = v_0 \cos\alpha$, $a_y = -g\cos\alpha$.

When particle lands it y co-ordinate becomes zero

$\Rightarrow \quad 0 = v_0 \cos\alpha T - \frac{1}{2} g\cos\alpha T^2$

$\Rightarrow \quad T = \frac{2v_0}{g}$

Now $x = v_0 \sin\alpha\, T + \frac{1}{2} g \sin\alpha\, T^2$

$$= v_0 \sin\alpha \frac{2v_0}{g} + \frac{1}{2} g\sin\alpha \left(\frac{2v_0}{g}\right)^2 = \frac{4v_0^2 \sin\alpha}{g}$$

***Example***: A batsman hits a ball at a height of 1.22 m above the ground so that ball leaves the bat an angle 45° with the horizontal. A 7.31 m high wall is situated at a distance of 97.53 m from the position of the batsman. Will the ball clear the wall if its maximum horizontal distance from the point of projection is 106.68 m. Take g=10 m/s$^2$.

***Solution:*** $\because \quad$ R(range) $= \frac{v_0^2 \sin 2\theta}{g}$

$\Rightarrow \quad v_0^2 = \frac{Rg}{\sin 2\theta} = Rg$ as $\theta = 45°$

$\Rightarrow \quad v_0 = \sqrt{Rg}$

Equation of trajectory

$Y = x\tan 45° - \frac{gx^2}{2v_0^2 \cos^2 45°}$

$= x - \frac{gx^2}{2Rg\frac{1}{2}}$

Putting $x = 97.53$, we get

$$y = 97.53 - \frac{10 \times (97.53)^2}{106.68 \times 10} = 8.35$$

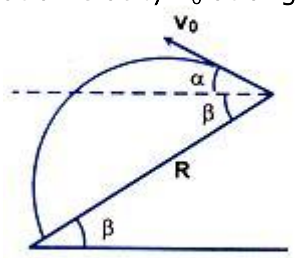
Hence height of the ball from the ground level is

$h = 8.35 + 1.22 = 9.577\ m$

as height of the wall is 7.31 m so the ball will clear the wall.

### 2.3.2.3 Motion down the plane

Let the particle is thrown at a velocity $v_0$ at angle 'α' with the horizontal as shown in figure.




$v_0$ sin (α +β)T- 1/2 gcosβ$T^2$=0 [for y'=0]

=> T = (2$v_0$ sin(α +β))/gcosβ

R = $v_0$ cos(α + β)T+ 1/2 g sin β$T^2$= ($v_0^2$)/g [(sin(2α +β)+sinβ)/(1-$\sin^2$β)]

Since α is the variable and maximum value of sin function is 1, therefore for R to be maximum, sin (2α +β)=1
and $R_{max}$ =( $v_0^2$)/g [(1+sinβ)/(1-$\sin^2$β)]= ($v_0^2$)/(g(1-sinβ)) down the plane.

## 2.4. UNIFORM CIRCULAR MOTION

### 2.4.1 Circular motion

Circular motion is rotation along a circle: a circular path or a circular orbit. It can be uniform, that is, with constant angular rate of rotation, or non-uniform, that is, with a changing rate of rotation.

Examples of circular motion are: an artificial satellite orbiting the Earth in geosynchronous orbit, a stone which is tied to a rope and is being swung in circles (cf. hammer throw), a racecar turning through a curve in a race track, an electron moving perpendicular to a uniform magnetic field, a gear turning inside a mechanism.

Circular motion is accelerated even if the angular rate of rotation is constant, because the object's velocity vector is constantly changing direction. Such change in direction of velocity involves acceleration of the moving object by a centripetal force, which pulls the moving object towards the center of the circular orbit. Without this acceleration, the object would move in a straight line, according to Newton's laws of motion.

### 2.4.2 Uniform circular motion

Uniform circular motion can be described as the motion of an object in a circle at a constant speed. As an object moves in a circle, it is constantly changing its direction. At all instances, the object is moving tangent to the circle. Since the direction of the velocity vector is the same as the direction of the object's motion, the velocity vector is directed tangent to the circle as well.

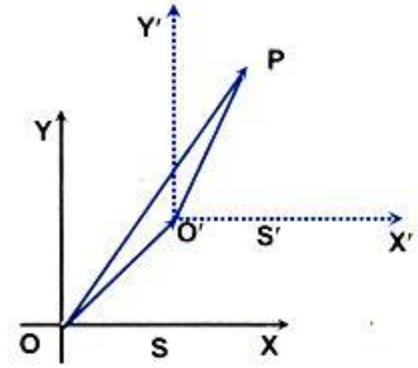
The final motion characteristic for an object undergoing uniform circular motion is the net force. The net force acting upon such an object is directed towards the center of the circle. The net force is said to be an inward or *centripetal* force. Without such an inward force, an object would continue in a straight line, never deviating from its direction. Yet, with the inward net force directed perpendicular to the velocity vector, the object is always changing its direction and undergoing an inward acceleration.

## 2.5 RELATIVE MOTION

### 2.5.1 Relative Velocity

Velocity of one object with respect to another or some reference point is called its relative velocity. The position, velocity and acceleration of a particle depend on the reference frame chosen.

A particle P is moving and is observed from two frames S and S'. The frame S is stationary and the frame S' is in motion.
Let at any time position vector of the particle P with respect to S is




$\overrightarrow{OP} = \vec{r}_{p,s}$ *and with respect to* $S'$ *is* $\overrightarrow{O'P} = \vec{r}_{p,s'}$

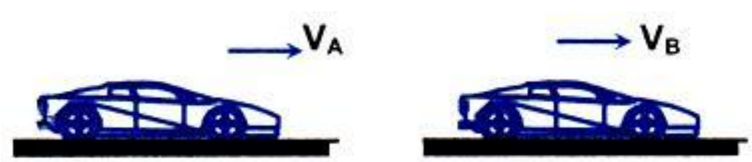
Position vector of the origin of S' with respect to S is $\overrightarrow{O'P} = \vec{r}_{p,s'}$
From vector triangle OO'P, we get
$\overrightarrow{O'P} = \overrightarrow{OP} - \overrightarrow{OP}'$

$$\Rightarrow \vec{r}_{p,s'} = \vec{r}_{p,s} - \vec{r}_{s',s} \Rightarrow \frac{d}{dt}(\vec{r}_{p,s'}) = \frac{d}{dt}(\vec{r}_{p,s}) - \frac{d}{dt}(\vec{r}_{s',s})$$

$$\Rightarrow \vec{v}_{p,s'} = \vec{v}_{p,s} - \vec{v}_{s',s} \Rightarrow \vec{v}_{p,s'} = \vec{v}_{p(absolute)} - \vec{v}_{s'(absolute)}$$

### 2.5.2 Physical Significance of Relative Velocity

Let two cars move unidirectionally. Two persons A and B are sitting in the vehicles as shown in figure. Assume, $V_A$ =10 m/s & $V_B$ = 4 m/s. The person A notices person B to be moving towards him with a speed of (10-4) m/s = 6 m/sec. That is the velocity of B with respect to (or relative to) A. That means $\vec{v}_{BA}$ is directed from B to A.

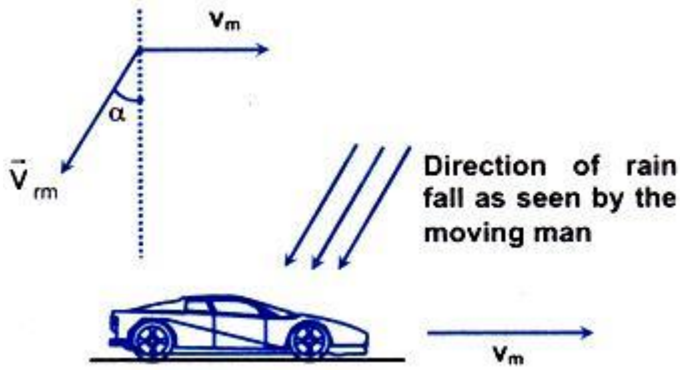


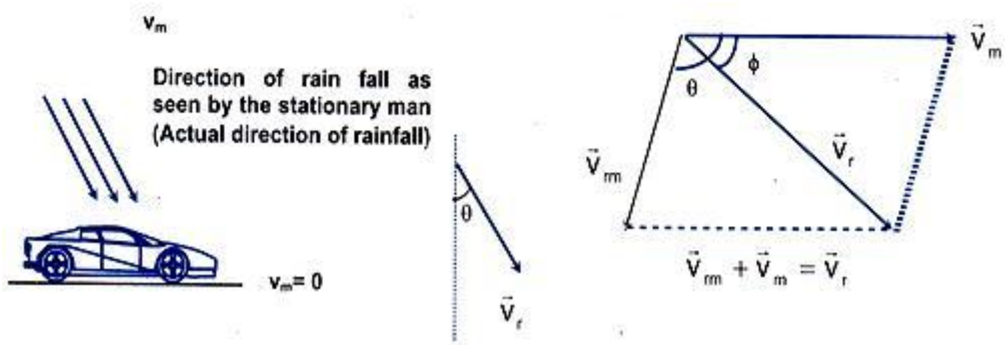

Similarly A seems to move towards B with a speed 6 m/sec. That means the velocity of A relative to B ($\vec{v}_{BA}$) has the magnitude 6 m/sec & directed from A to B as shown in the figure.

$$\Rightarrow \vec{v}_{AB} = -\vec{v}_{BA}$$

$$In\ general, \vec{v}_{AB} = \vec{v}_B - \vec{v}_A$$
$$\Rightarrow\ |\vec{v}_{BA}| = |\vec{v}_{AB}|$$
$$v_{AB} = \sqrt{v_A^2 + v_A^2 - 2v_A v_B \cos\theta}$$
$$and\ \theta = \tan^{-1}\left\{\frac{v_B \sin\theta}{(v_A - v_B\cos\theta)}\right\}$$

**2.5.3 Relative Motion between Rain and Man**




***Example***: A man walking eastward at 5 m/s observes that wind is blowing from the north. On doubling his speed eastward, he observes that wind is blowing from north-east. Find the velocity of the wind.

**Solution:** Let velocity of the wind is

$$\vec{v}_w = (v_1\hat{\imath} + v_2\hat{\jmath})m/s$$

And velocity of the man is

$$\vec{v}_m = 5\hat{\imath}$$

$$\therefore\ \vec{v}_{wm} = \vec{v}_w - \vec{v}_m = (v_1 - 5)\hat{\imath} + v_2\hat{\jmath}$$

In first case,

$v_1 - 5 = 0$ => $v_1 = 5$ m/s.

In the second case, $\tan 45^o = v_2/(v_1 - 10)$

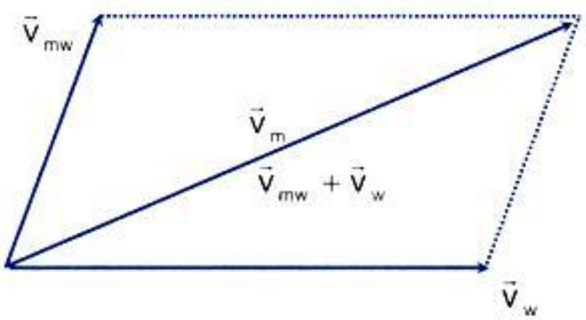
=> $v_2 = v_1 - 10 = -5$ m/s.

=> $\vec{v}_w = (5\hat{\imath} - 5\hat{\jmath})\ m/s$

### 2.5.4 Relative Motion of a Swimmer in Flowing Water

Take $\vec{v}_m$ = velocity of man

$\vec{v}_w$ = velocity of flow of river,




$\vec{v}_{wm}$ = velocity of swimmer w.r.t. river

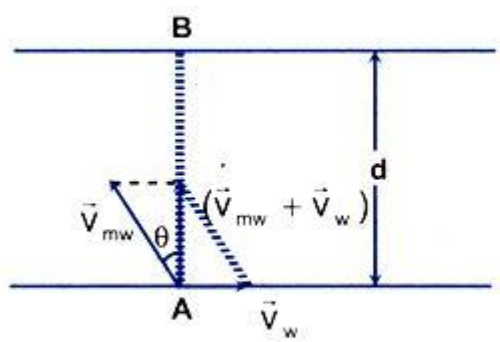
$\vec{v}_m$ can be found by the velocity addition of $\vec{v}_{wm}$ and $\vec{v}_w$.

$$\vec{v}_{wm} = \vec{v}_w - \vec{v}_m$$

Crossing of the River with Minimum Drift

**Case 1:** $\vec{v}_{wm} > \vec{v}_w$

A man intends to reach the opposite bank at the point directly opposite to the stationary point. He has to swim at angle θ with a given speed $\vec{v}_{wm}$ w.r.t. water, such that his actual velocity $\vec{v}_m$ will direct along AB, that is perpendicular to the bank (or velocity of water $\vec{v}_w$).



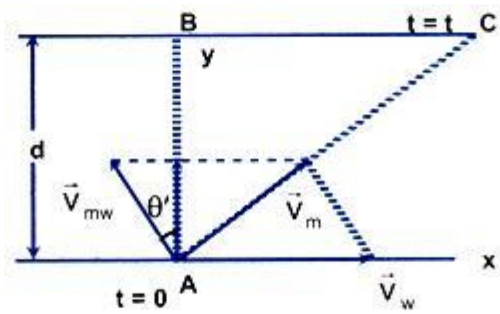

=> For minimum drift, $\vec{v}_m \perp \vec{v}_w$

Observing the vector-triangle $v_w = v_{mw} \sin\theta$ & $v_m = v_{mw} \cos\theta$
=> $\theta = \sin^{-1}(v_w/v_{mw})$ & $v_m = \sqrt{((v_{mw})^2 - (v_w)^2)}$
=> The time of crossing, $t = d/v_m$
=> $t = d/\sqrt{(v_{mw})^2 - (v_w)^2)}$

**Case 2** : $v_w > v_{mw}$

Let the man swim at an angle θ' with normal to the bank for minimum drift. Suppose the drift is equal to zero. For zero drift, the velocity of the man along the bank must be zero.

=> $v_m = v_w - v_{mw} \sin\theta' = 0$




This gives, $\sin\theta' = v_w / v_{mw}$, since $v_w > v_{mw}$, $\sin\theta' > 1$ which is impossible. Therefore, the drift cannot be zero.

Now, let the man swim at an angle θ with the normal to the bank to experience minimum drift. Suppose that the drifting of the man during time t when he reaches the opposite bank is

BC = x

$x = (v_m)_x (t)$ ... (1)

where $t = AB/((v_m)_y \cos\theta) = d/(v_{mw} \cos\theta)$ ... (2)

and $(v_m)_x = v_w - v_{mw} \sin\theta$ ... (3)

Using (1), (2) & (3), we obtain

$x = (v_w - v_{mw} \sin\theta \;\; d/(v_{mw} \cos\theta))$

$= (v_w/v_{mw} \sec\theta - \tan\theta)d$

$x = (v_w/v_{mw} \sec\theta - \tan\theta)d$ ... (4)

For x to be minimum,

$dx/d\theta = (v_w/v_{mw} \sec\theta - \tan\theta - \sec^2\theta)d = 0$

$v_w/v_{mw} \tan\theta = (\sec\theta)$ => $\sin\theta = v_{mw}/v_w$

$\theta = \sin^{-1}(v_{mw}/v_w)$

Substituting the value of θ in (4), we obtain

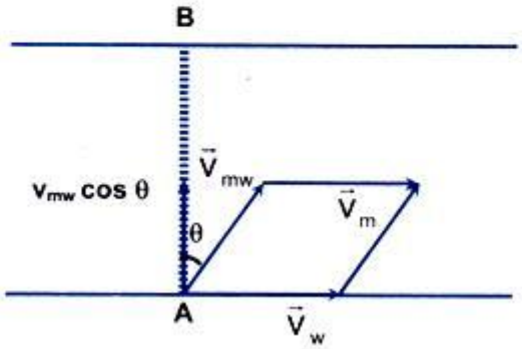
$$x = \left[\frac{\sqrt{v_w^2 - v_{mw}^2}}{v_{mw}}\right] d$$

### 2.5.5 Crossing of the River in Minimum Time

**Case 1: To reach the opposite bank for a given $v_{mw}$**

Let the man swim at an angle *θ* with AB. We know that the component of the velocity of man along shore is not responsible for its crossing the river. Only the component of velocity of man ($v_m$) along AB is responsible for its crossing along AB.

The time of crossing = $t = AB/(v_{mw} \cos\theta)$




Time is minimum when cos θ is maximum

The maximum value of cos θ is 1 for θ = 0.

That means the man should swim perpendicular to the shore

=> $\vec{v}_{wm} \perp \vec{v}_w$

=> Then $t_{min}$ = $d/(v_{mw}\cos\theta)|(\theta=0) = d/v_{mw}$ => $t_{min} = d/v_{mw}$

**Case 2:**

To reach directly opposite point on the other bank for a given $v_{mw}$ & velocity v of walking along the shore.

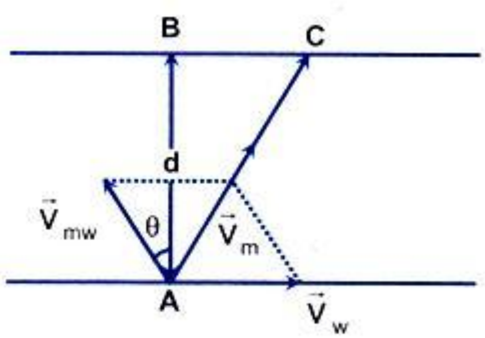
To attain the direct opposite point B in the minimum time. Let the man swim at an angle θ with the direction AB. The total time of journey t = the time taken from A to C and the time taken from C to B

=> $t = t_{AC} + t_{CB}$

where $t_{AC} = AB/v_{mv}\cos\theta$ & $t_{CB} = BC/v$ where v = walking speed of the man from C to B.

=> $t = AB/v_{mv}\cos\theta + BC/v$

Again $BC = (v_m)_x t$




=> $BC = (v_w - v_{mw}\sin\theta)(AB/v_{mw}\cos\theta)$

Using (1) & (2) we obtain,

$t = AB/v_{mw}\cos\theta + ((v_w - v_{mw}\sin\theta)/v(v_{mv}\cos\theta))$

=> $t = AB[(1+v_w/v)\sec\theta/v_{mv} - \tan\theta/v]$

=> $t = d/v_{mv}[(1+v_w/v)\sec\theta/v_{mv} - \tan\theta/v]$

Putting $dt/d\theta = 0$, For minimum t we get

$dt/d\theta = d/d\theta[d/v_{mv}\,(1+v_w/v)\sec\theta/v_{mv} - \tan\theta/v]$

$= [\sec\theta/v_{mv} - \tan\theta/v\ (1+v_w/v)\ (\sec^2\theta)/v] = 0$

=> $\tan\theta/v_{mv}\ (1+v_w/v)\ \sec\theta/v$

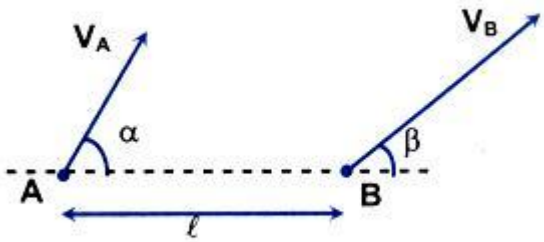
=> $\sin\theta = (v_{mw}/v+v_w)$

=> $\theta = \sin^{-1}(v_{mw}/v+v_w)$

This expression is obviously true when $v_{mw} < v + v_w$

**2.5.6 Velocity of Separation/Approach, Relative Angular Velocity**

Let there be two particles A and B with velocity $\vec{v}_A$ and $\vec{v}_B$ at any instant as visualized from ground frame.




If we visualize the motion of B from frame of A the velocity of particle B would be $\vec{v}_B - \vec{v}_A$.

If α, β be the angle made with line AB.
Then $V_B \cos\beta - V_A \cos\alpha$ is relatively velocity of B w.r.t. A along line AB.

- If $V_B \cos\beta - V_A \cos\alpha > 0$; it is called as velocity of separation.
- If $V_B \cos\beta - V_A \cos\alpha < 0$; it is called as velocity of approach.

$V_B \sin\beta - V_A \sin\alpha$ is relative velocity of B w.r.t. A along direction perpendicular to AB. If length of AB is l.

Then, angular velocity B w.r.t. A is $(V_B \sin\beta - V_A \sin\alpha)/l$
Relative angular velocity = $(V_B \sin\beta - V_A \sin\alpha)/l$.