Gravitation

Contents

6. GRAVITATION

6.1 NEWTON'S LAW OF GRAVITATION

Newton's Law of Gravitation states that "Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them".

Consider two bodies A and B of masses m_A and m_B, attracting each other with forces $\vec{F}_{AB}$ (force on A due to B) and $\vec{F}_{BA}$ (force on B due to A), respectively.

Then, $\vec{F}_{AB} = -\vec{F}_{BA}$

$\vec{F}_{AB} = |\vec{F}_{AB}|$ = magnitude of the attractive force

$= G(m_A\, m_B)/d^2$

where d is the distance between them. G is a universal constant known as Universal Gravitational constant. Its value was first measured by Cavendish and is now known to be:

$G = 6.62726 \times 10^{-11}$ N-m^2/kg^2

"The space around a body within which its force of gravitational attraction is perceptible (by any other body in this space) is called its gravitational field."

The intensity E, of the gravitational field of a mass 'm' at a point at distance 'r' from it is the force experienced by a unit mass placed at this point in the field. (Assuming that the presence of unit mass does not affect the gravitational field of the mass m)

Thus, $E = (-m/r^2)\, G$

The negative sign is because the intensity of the field is directed towards the mass and away from the unit mass.

If $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ ...... are the gravitational forces acting on the particle A due to particles P_1, P_2, P_3, ..., respectively, then net gravitational force on a particle A due to particles P_1, P_2, ..., P_n is given by:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots\ldots.. + \vec{F}_n$$

Example: Two Lead balls of radius 10 cm and 1 cm are placed with their centres 1 metre apart. Calculate the force of attraction between them. The density of Lead is 5.51×10^3 kg/m^3.

Solution: Density of balls, $\rho = 5.51 \times 10^3$ kg/m^3

Force of attraction between them, $F = Gm_1m_2/r^2$

where $m_1 = 4/3\Pi r_2^3 \times \rho = 5.51 \times 10^3 \times 4/3\ \Pi 10^{-3} \approx 23$ kg

and $m_2 = 4/3\Pi r_2^3 \times \rho \approx 23 \times 10^{-3}$ kg

$G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$

$\therefore \quad F = 6.67 \times 10^{-11} \times 21 \times 21 \times 10^{-3} N = 3.5 \times 10^{-11}\ N$

Note: 1. The gravitational force is an attractive force.

2. The gravitational force between two particles does not depend on the medium.

3. The gravitational force between two particles is along the straight line joining the particles (called line of centers).

Example: Three identical particles, each of mass m, are placed at the three corners of an equilateral triangle of side 'a'. Find the force exerted by this system on another particle of mass m placed at

(a) the mid-point of a side

(b) the centre of the triangle

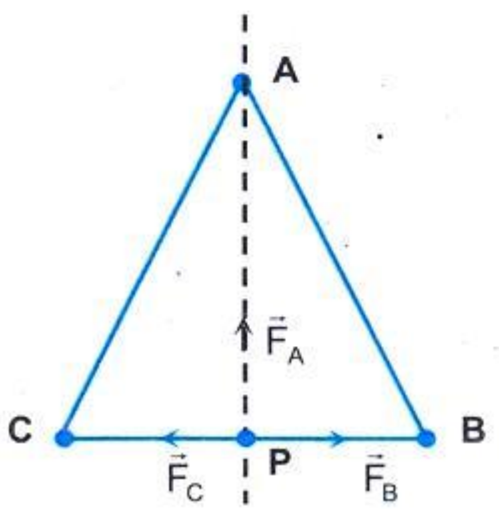
Solution: (a) Using the principle of superposition

$$\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C$$

When the particle is placed at the mid point of a side (at P), $\vec{F}_C = -\vec{F}_B$, and they cancel each other.

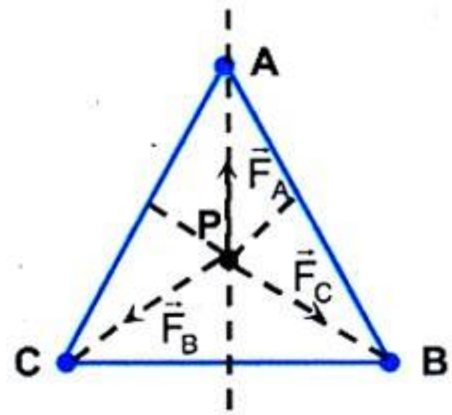
Hence, force experienced by the particle, $\vec{F} = \vec{F}_A$

$|\vec{F}| = |F_A| = Gmm/(AP)^2 = Gm^2/(a \sin 60^0)^2$



(b) If the particle is placed at the centre of the triangle, the net force on the particle P due to particles placed at the corners A, B and C will be zero.

Hence, $\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C = 0$.



6.2 ACCELERATION DUE TO GRAVITY (g)

The Earth attracts a mass m on its surface by a force F given as:

$$F = GM_e\ m/R_e^2$$

where M_e is the mass of the Earth and R_e its radius.

This force imparts acceleration to the mass m, which is known as acceleration due to gravity (g).

By Newton's second law, acceleration = Force/ Mass

$$\Rightarrow g = F/m = GM_e/R_e^2$$

Example: Find the value of g at the surface of Earth?

Radius of Earth = 6.37×10^6 meter.

Mass of Earth = 6×10^{24} kg.

Solution: As $g = GM/R^2$ ...... (1)

$G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$

$M = 6 \times 10^{24}$ kg

$R = 6.37 \times 10^4$ m

Put all values in (1), we get $g = 9.8\ m/s^2$

6.2.1 Variation of Acceleration due to Gravity (g)

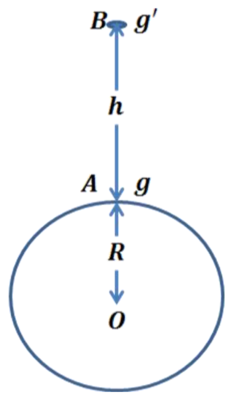
i) Due to altitude

Consider a mass m at a height h from the surface of the earth.

Now, the force acting on the mass due to gravity is

$$F = GMm/(R+h)^2$$

where M is the mass of the earth and R is the radius of the earth.



If the acceleration due to gravity at the given height is g', then

$$F = mg' = G\,\frac{Mm}{(R+h)^2}.$$

$$\Rightarrow \quad g' = G\,\frac{M}{R^2(1+h/R)^2} = \frac{GM}{R^2}\times\left(1+\frac{h}{R}\right)^{-2} = g\left(1-\frac{2h}{R}\right)(h \ll R)$$

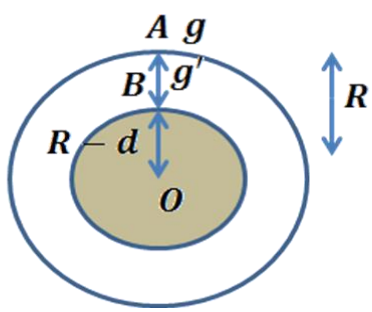
(Expanding binomially and neglecting the higher order terms).

(ii) Due to depth:

If a particle of mass m is kept at a depth 'd' from the surface of earth, then gravitational force exerted on the particle of mass 'm'.

$$mg' = (GM'\,m)/(R-d)^2$$

$$g' = GM(R-d)/R^3$$



where M' = mass of earth within radius of (R - d)

$\therefore \quad g' = (GM/R^3)(R-d) = (GM/R^3)\,R(1-d/R)$

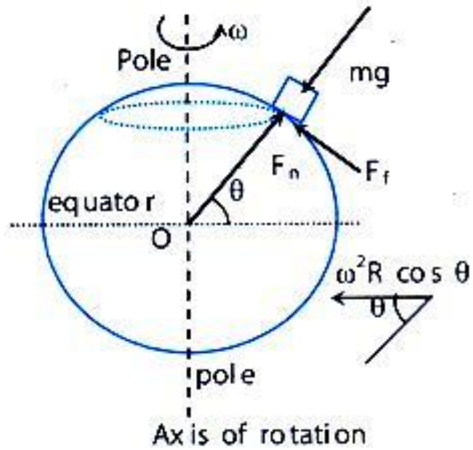
$\quad\quad = (GM/R^2)\,(1-d/R)$

$\quad g' = g(1-d/R)$

(iii) Due to rotation of the earth:

Consider a body at a point with altitude θ, on the surface of the earth.

Let R = radius of the earth and ω = angular velocity of the earth about its own axis.



Acceleration of the body with respect to the earth's centre O is $(R\cos \theta)\omega^2$ directed towards the axis of rotation (i.e. the centripetal acceleration).

From Newton's second law in the radial direction

$$mg - F_n = m(R \cos \theta)\, \omega^2 \cos \theta$$

or $F_n = m[g - R\omega^2 \cos^2 \theta]$

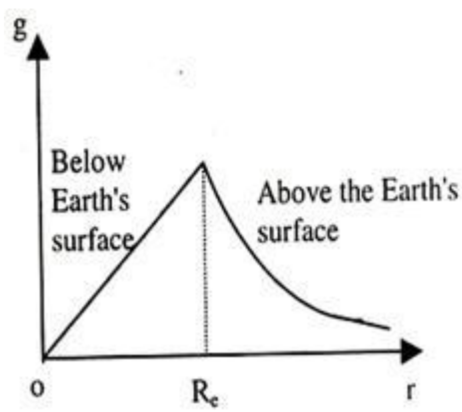
or $F_n/m = g' = g - R\omega^2 \cos^2 \theta$

where g' is the apparent value of the acceleration

At poles, $\theta = 90^\circ \Rightarrow g' = g$

At the equator, $\theta = 0^\circ \Rightarrow g' = g - R\omega^2$

Figure given below has illustrated the variation of g with the distance of separation from Earth's centre.



Example: Find the value of g at a height equal to the radius of Earth.

Solution: $g = GM/(R+h)^2$ and $g_0 = GM/R^2$ (At the surface of Earth).

$g/g_0 = R^2/(R+h)^2 = R^2/4R^2 \quad \because h = R$

$\Rightarrow g = g_0/4 = 9.8/4 = 2.45\ m/s^2$.

Note: Here h = R, so we cannot apply $g' = g(1-2h/R_e)$

Example: At what angular velocity Earth should rotate, for the weight of an object at the equator to be zero? What would be the duration of a day in this case? (Radius of Earth = 6.4×10^6 m and $g_0 = 9.8$ m/s^2)

Solution: For the weight to be zero, the value of g should be zero. That is

Here, $g' = g_0 - R_e\omega^2 = 0$

or $\omega = \sqrt{(g_0/R_e)} = \sqrt{(9.8/(6.4\times10^6))} = 1.2 \times 10^{-3}$ rad/s

The duration of one day will be equal to the time period of rotation

$T = 2\Pi/\omega = 2\Pi/(1.2\times10^{-3})$ sec.

6.3 KEPLER'S LAWS OF PLANETARY MOTION

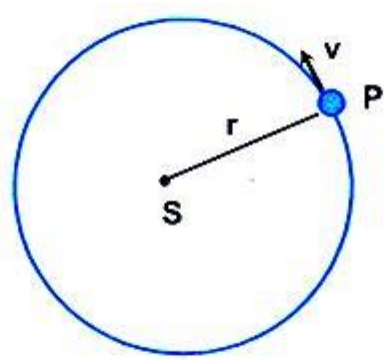
Johannes Kelper discovered three empirical laws by using the data on planetary motion.

1. Each planet moves in an elliptical orbit, with the sun at one foci of the ellipse.
2. A line from the sun to a given planet sweeps out equal areas in equal intervals of time.
3. The square of the periods of the planets are proportional to cube of their mean distance (or semi-major axis) from the sun.

These laws go by the name 'Kepler's laws of planetary motion'. It was in order to explain the origin of these laws, among other phenomena, that Newton proposed the theory of gravitation.

Consider a planet of mass m rotating around the sun (mass M >> m) in a circular orbit of radius r with velocity v. Then, by applying Newton's law of gravitation and the second law of motion, we can write

Gravitational force = mass × centripetal acceleration



i.e. $GMm/r^2 = m(v^2/r)$ ... (1)

or, $v^2 = GM/r$ ... (2)

As the moment of the gravitational force about S is zero, the angular momentum of the planet about the sun remains constant. This is the meaning of Kepler's 2nd law of motion.

The time period of rotation, T, of the planet around the sun is given by,

$T = 2\Pi r/v = 2\Pi r/\sqrt{GM/r} = (2\Pi/\sqrt{GM})\, r^{3/2}$ ... (3)

Squaring both sides,

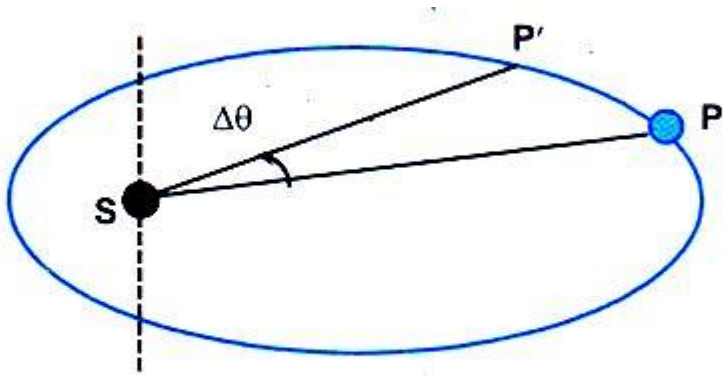
$T^2=(4\Pi^2/GM)r^3$ ... (4)

which is Kepler's 3rd law of motion.

Note: The constant of proportionality in the above equation depends only on the mass of the sun (M) but not on the mass of the planet. Kepler's Laws are also valid for the motion satellites around the earth.

6.3.1 Kepler's Second Law

Consider a planet P that moves in an elliptical orbit around the sun, and let P and P' be the positions of the planet at time t and t + Δt (where Δt is a very small time interval). If the angular displacement of the planet is Δθ, then the area swept out by the line joining the planet and sun (SP) in time Δt is:



ΔA = area of the section SPP'

$= (1/2)\, r^2.\Delta\theta;$

where r = the length SP.

The area velocity,

$v_A = \Delta A/\Delta t = (½)r^2\Delta\theta/\Delta t$

$= (½)\, r^2\omega = \text{constant}$ .... (5)

Hence, the angular momentum of the planet does not vary, i.e. the areal velocity of the planet remains constant. At its aphelion (farthest point from the sun, r is large), the planet moves slowly and at its perihelion (nearest point from the sun, r is small) the planet moves fastest.

Example: Calculate the mass of the Sun from the following data; distance between the Sun and the Earth = 1.49×10^{11} m, G = 6.67×10^{-11} SI units and one year = 365 days.

Solution: Force of attraction between the sun and the earth = Gm_sm_E/d_{SE}^2

Considering the orbit of the earth as nearly circular, the centripetal force acting on the earth is $m_E\, d_{SE}\omega^2$.

=> $m_E\, d_{SE}\, \omega^2 = Gm_sm_E/d_{SE}^2$

$m_S = d_{SE}^3.\omega^2/G = 4\Pi^2 d_{SE}^2/GT^2$

$=(4\times(3.14)^2\times(1.49\times10^{11})^3)/(6.67\times10^{-11}\times(365\times24\times60\times60)^2)$

$=1.32 \times 10^{19}$ kg.

Example: A Saturn year is 29.5 times the earth year. How far is Saturn from the sun (M) if the earth is 1.5×10^8 km away from the sun?

Solution: Given: $T_S = 29.5\ T_e$; $R_e = 1.5 \times 10^{11}$ m

Now, according to kepler's third law

$T_S^2/T_e^2 = R_s^3/R_e^3$

$R_S=R_e(T_S/T_e)^{2/3}$

$=1.5\times10^{11}\ ((29.5T_e)/T_e)^{2/3}=1.43\times10^{12}$ m

$=1.43\times10^9$ km

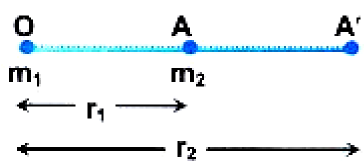
6.4 GRAVITATIONAL FIELD AND INTENSITY

The space around a body where the gravitational force exerted by it can be experienced by any other particle is known as the gravitational field of the body. The strength of this gravitational field is referred to as intensity, and it varies from point to point.

6.5 GRAVITATIONAL POTENTIAL ENERGY

Gravitational potential energy is energy an object possesses because of its position in a gravitational field. The most common use of gravitational potential energy is for an object near the surface of the Earth where the gravitational acceleration can be assumed to be constant at about 9.8 m/s^2

Change in gravitational potential energy (U) of a system is defined as the negative of the work done by the gravitational force as the configuration of the system is changed.



Change in gravitational potential energy of two point masses m_1 and m_2 as their separation is changed from r_1 to r_2 is given by

$U(r_2) - U(r_1) = Gm_1m_2\ [1/r_1 - 1/r_2]$

If, at infinite separation, gravitational potential energy is assumed to be zero, then the gravitational potential energy of the above two point mass system at separation r,

$U(r) = -Gm_1m_2/r$

Example: What is the gravitational Potential energy of the Moon-Earth system, relative to the potential energy at infinite separation?

Solution: M = Mass of Earth = 5.98×10^{24} kg m = mass of Moon = 7.36 = 10^{22} k

d = mean separation between Earth and moon = 3.82×10^8 m.

∴ Gravitational potential energy of the Moon-Earth system (U)

= -GMm/d

Put all values U = -7.68×10^{28} J

6.6 GRAVITATIONAL POTENTIAL

The gravitational potential at a point is equal to the work by the external force as a particle of unit mass is brought from infinity to its position in the gravitational field.

Gravitational field around a material body can be described not only by gravitational intensity vector $\vec{E}$ but also by a scalar function, the gravitational potential V. The gravitational potential at any point may be defined as the potential energy per unit mass of a test mass placed at that point.

V = U/m

(where U is the gravitational potential energy of the test mass m).

Thus, if the reference point is taken at infinite distance, the potential at a point in the gravitational field is equal to amount of work done by the external agent per unit mass in bringing a test mass from infinite distance to that point. The expression for the potential is given by

$$V = \int_{\infty}^{P} \vec{E}.d\vec{r}$$

With the above definition, the gravitational potential due to a point mass M at a distance r from it is

$$V = -\int_{\infty}^{r} \frac{GM}{r^2}\hat{r}.d\vec{r} = \int_{\infty}^{r} \frac{GM}{r^2}dr = -\frac{GM}{r}$$

Potential is a scalar quantity. Therefore, at a point in the gravitational field of a number of material particles, the resultant potential is the arithmetic sum of the potentials due to all the particles at that point. If masses m_1, m_2, ..., m_n are at distances r_1, r_2, r_3, ..., r_n then potential at the given point is

$V = -G(m_1/r_1 + m_2/r_2 + m_3/r_3 + ...)$

The field and the potential are related as, E = -dV/dr

6.6.1 Gravitational potential due to a shell

(i) At a point outside the shell is: -GM/r (r>R)

(ii) At a point on the surface of the shell is: -GM/R

(iii) At a point inside the shell is: -GM/R

6.6.2 Gravitational potential (V) due to a uniform solid sphere

(i) Outside of the sphere at a distance r form the centre, $V = -GM/r$.

(ii) Inside the sphere at a distance r from the centre, $V = -GM/R^3 (R^2/2 - r^2/6)$

6.6.3 Binding Energy

Binding energy of a system of two bodies is the amount of minimum energy needed to separate the bodies to a large distance.

If two particles of masses m_1 and m_2 are separated by a distance r, then the gravitational potential energy of the system is given by

$$U = Gm_1m_2/r$$

Let T amount of energy is supplied to the system to separate the bodies by a large distance. When the bodies are separated by a large distance, gravitational potential energy of the system is zero. For minimum T, conserving energy for initial and final positions,

$$T + U = 0$$

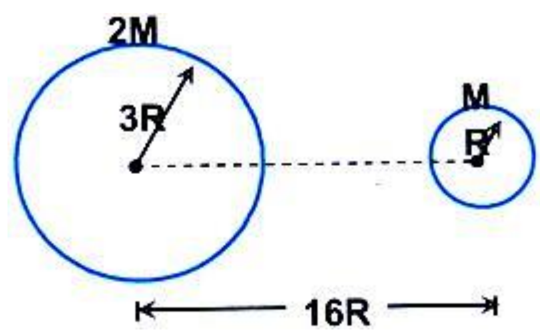
$$\Rightarrow \quad T - Gm_1m_2/r = 0 \text{ or } T = Gm_1m_2/r$$

Hence, binding energy of a system of two particles separated by a distance r is

$$T = Gm_1m_2/r,$$

where m_1 and m_2 are the masses of the particles.

Example: Two spherical bodies of masses 2M and M and of radii 3R and R, respectively, are held at a distance 16R from each other in free space. When they are released, they start approaching each other due to the gravitational force of attraction. Then, find:



(a) the ratio of their acceleration during their motion.

(b) their velocities at the time of impact.

Solution: (a) Due to the mutual attraction, the masses attract each other.

If the acceleration are a_1 and a_2,

the net external force on the system = 0

$\Rightarrow \quad \vec{a}_{cm} = 0 \Rightarrow m_1a_1 - m_2a_2 = 0$

or $a_1/a_2 = m_2/m_1 = 1/2$

6.7 ESCAPE VELOCITY

Escape velocity is defined as the smallest speed that we need to give an object in order to allow it to completely escape from the gravitational pull of the planet on which it is sitting.

The object must have greater energy than its gravitational binding energy to escape the earth's gravitational field. So:

$$1/2\ mv^2 = GMm/R$$

Where m is the mass of the object, M mass of the earth, G is the gravitational constant, R is the radius of the earth, and v is the escape velocity. It simplifies to:

$$v = \sqrt{(2GM/R)}$$

$$\text{or } v = \sqrt{(2gR)}$$

Where g is acceleration of gravity on the earth's surface.

The value evaluates to be approximately:

11100 m/s
40200 km/h
25000 mi/h

So, an object which has this velocity at the surface of the earth, will totally escape the earth's gravitational field (ignoring the losses due to the atmosphere.).

6.8 ORBITAL VELOCITY

Orbital Velocity is the velocity which is given to an artificial earth's satellite a few hundred kilometers above the earth's surface so that it may start revolving round the earth. It is denoted by V_o.

6.8.1 Expression for orbital velocity

Let, m = Mass of satellite
r = radius of circular orbit of satellite
h = height of satellite above surface of earth.
R = radius of earth
V_o = orbital velocity
M = Mass of the earth.

We know

$$\text{Centripetal force} = \frac{mv^2}{r}$$

Centripetal force required by the satellite to keep moving in a circular orbit is produced by the gravitational force between the satellite and the earth. Therefore,

$$\frac{mV_o^2}{r} = \frac{GMm}{r^2}$$

Here r=R+h

Let g_h be the acceleration due to gravity at heigh h

Therefore, weight of satellite = mg_h

Force of attraction between body (Satellite) and earth must be balance by weight of satellite.

$$mg_h = \frac{GMm}{(R+h)^2}$$

$$GM = g_h (R+h)^2$$

$$V_0^2 = \frac{GM}{R+h}$$

Remember, rockets are used to move the body to required height and we can place space shuttle or satellites in earth's orbit.

6.9 TIME PERIOD OF SATELLITE

Time taken by a satellite to complete one revolution is called its time period.

i.e. $$T = \frac{2\pi r}{V_0}$$

or $$T^2 = \frac{4\pi^2 (R+h)}{V_0}$$

where r = R+h

$$T^2 = \frac{4\pi^2 (R+h)^2}{\frac{GM}{R+h}} = \frac{4\pi^2 (R+h)^3}{GM}$$

If time period of a satellite is 24 hrs. Then

$r = [GMT^2/4\Pi^2]^{1/3}$ = 42400 km and h = 36000 km.

This gives the height of a satellite above the Earth's surface whose time period is same as that of Earth's. Such a satellite appears to be stationary when observed from the Earth's surface and is hence known as Geostationary satellite.

6.10 GEOSTATIONARY SATELLITE

A geostationary satellite is any satellite which is placed in a geostationary orbit. Satellites in geostationary orbit maintain a constant position relative to the surface of the earth.

Example: An artificial satellite of mass 100 kg is in circular orbit at 500 km above the earth's surface. Take the radius of the earth as 6.5×10^6 m. Find the acceleration due to gravity at any point along the satellite path.

Solution: Here, h = 500 km = 0.5×10^6 m

$R = 6.5 \times 10^6$ m

$r = R + h = 6.5 \times 10^6 + 0.5 \times 10^6 = 7.0 \times 10^6$ m

$g' = g(R/R+h)^2 = 9.8((6.5 \times 10^6)/(7.0 \times 10^6)) = 8.45$ m/s^2

6.11 SOLVED EXAMPLES

Example 1: The time period of Moon around the Earth is n times that of Earth around the Sun. If the ratio of the distance of the Earth from the Sun to that of the distance of Moon from the Earth is 392, find the ratio of mass of the Sun to the mass of the Earth. (Assume that the bodies revolve in circular orbits)

Solution: The time period T_e of Earth around Sun of mass M_s is given by

$$T_e^2 = 4\Pi^2/GM_s \times r_e^3, \qquad \text{...... (1)}$$

where r_e is the radius of the orbit of Earth around the Sun.

Similarly, time period T_m of Moon around Earth is given by

$$T_m^2 = 4\Pi^2/GM_e \times r_m^3, \qquad \text{...... (2)}$$

where r_m is the radius of the orbit of Moon around the Sun.

Dividing equation (1) by (2), we get

$(T_e/T_m)^2 = (M_e/M_s)\ (r_e/r_m)^3$

$\therefore \quad (M_s/M_e) = (T_m/T_e)^2 \times (r_e/r_m)^3 \qquad$ ...... (3)

Substituting the given values, we get

$(M_s/M_e) = (392)^3\ n^2$

Example 2: Imagine a planet whose diameter and mass are both one half of those of Earth. The day's temperature of this planet's surface reaches upto 800K. Find whether Oxygen molecules are possible in the atmosphere of this planet.

Solution: Escape velocity, $v_e = \sqrt{(2GM/R)}$

Let v_p = escape velocity on the planet.

v_e = escape velocity on the Earth.

$\therefore \quad v_p/v_e = \sqrt{(M_P/R_P \times R_e/M_e)} = \sqrt{(1/2\times 2/1)} = 1$

$\therefore \quad v_p = v_e = 11.2$ km/s.

From kinetic theory of gases

$v_{rms} = \sqrt{(3RT/M)} = \sqrt{(3NKT/M)} = \sqrt{(3NKT/Nm)}$

where N = Avogadro's number

m = mass of oxygen molecule

K = Boltzmann constant

$v_{rms} = \sqrt{(3RT/m)}$

$= \sqrt{((3\times 1.38\times 10^{-23}\times 800)/(5.3\times 10^{-26}))}$

(here $m = 5.3 \times 10^{-26}$ kg).

$v_{r.m.s} = 0.79$ km/s

As v_{rms} is very small compared to escape velocity on the planet, molecules cannot escape from the surface of the planet's atmosphere.