Trigonometrical Equations

An equation involving one or more trigonometrical ratio of an unknown angle is called a trigonometrical equation *i.e.*, $\sin x + \cos 2x = 1$; $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$; $|\sec(\theta + \pi/4)| = 2$ etc.

A trigonometric equation is different from a trigonometrical identity. An identity is satisfied for every value of the unknown angle. e.g., $\cos^2 x = 1 - \sin^2 x$ is true $\forall x \in R$ while a trigonometric equation is satisfied for some particular values of the unknown angle.

Roots of trigonometrical equation

The value of unknown angle (a variable quantity) which satisfies the given equation is called the root of an equation e.g., $\cos\theta = \frac{1}{2}$, the root is $\theta = 60^o$ or $\theta = 300^o$ because the equation is satisfied if we put $\theta = 60^o$ or $\theta = 300^o$.

Solution of trigonometrical equations

A value of the unknown angle which satisfies the trigonometrical equation is called its solution.

Since all trigonometrical ratios are periodic in nature, generally a trigonometrical equation has more than one solution or an infinite number of solutions. There are basically three types of solutions:

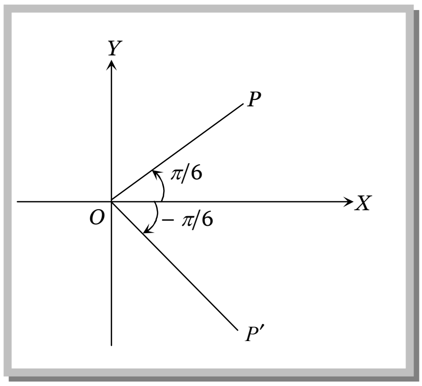
(1) **Particular solution :** A specific value of unknown angle satisfying the equation.

(2) **Principal solution :** Smallest numerical value of the uknown angle satisfying the equation (Numerically smallest particular solution.)

(3) **General solution :** Complete set of values of the unknown angle satisfying the equation. It contains all particular solutions as well as principal solutions.

When we have two numerically equal smallest unknown angles, preference is given to the positive value in writing the principal solution.

e.g. $\sec\theta = \frac{2}{\sqrt{3}}$ has $\pi/6, -\pi/6, 11\pi/6, -11\pi/6, 23\pi/6, -23\pi/6$ etc.



As its particular solutions out of these, the numerically smallest angles are $\pi/6$ and $-\pi/6$ but the principal solution is taken as $\theta = \pi/6$ to write the general solution we notice that the positive on P or P' can be obtained by rotation of OP or OP' around O through a complete angle (2π) any number of times and in any direction (clockwise or anticlockwise)

$\therefore$ The general solution is $\theta = 2k\pi \pm \pi/6, k \in Z$.

General solution of standard trigonometrical equations

(1) **General solution of the equation sin θ = sin α :** If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$; $n \in I$

Note : ❑ The equation $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ is equivalent to $\sin\theta = \sin\alpha$. So these two equation having the same general solution.

(2) **General solution of the equation cos θ = cos α :** If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$; $n \in I$

Note : ❑ The equation $\sec\theta = \sec\alpha$ is equivalent to $\cos\theta = \cos\alpha$, so the general solution of these two equations are same.

(3) **General solution of the equation tan θ = tan α** : If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha; n \in I$

Note : ❑ The equation $\cot\theta = \cot\alpha$ is equivalent to $\tan\theta = \tan\alpha$ so these two equations having the same general solution.

General solution of square of the trigonometrical equations

(1) **General solution of sin²θ = sin² α** : If $\sin^2\theta = \sin^2\alpha$ or, $2\sin^2\theta = 2\sin^2\alpha$ (Both the sides multiply by 2)

or, $1 - \cos 2\theta = 1 - \cos 2\alpha$ or, $\cos 2\theta = \cos 2\alpha$, $2\theta = 2n\pi \pm 2\alpha; n \in I$, $\theta = n\pi \pm \alpha; n \in I$.

(2) **General solution of cos² θ = cos² α** : If $\cos^2\theta = \cos^2\alpha$ or, $2\cos^2\theta = 2\cos^2\theta$ (multiply both the side by 2) or, $1 + \cos 2\theta = 1 + \cos 2\alpha$ or, $2\theta = 2n\pi \pm 2\alpha$, $\theta = n\pi \pm \alpha; n \in I$.

(3) **General solution of tan² θ = tan²α** : If $\tan^2\theta = \tan^2\alpha$ or, $\frac{\tan^2\theta}{1} = \frac{\tan^2\alpha}{1}$ (Using compo.and divid. rule)

$\frac{\tan^2\theta + 1}{\tan^2\theta - 1} = \frac{\tan^2\alpha + 1}{\tan^2\alpha - 1}$ or, $\frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{1 + \tan^2\alpha}{1 - \tan^2\alpha}$ or, $\frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$ or, $\cos 2\theta = \cos 2\alpha$, $\theta = n\pi \pm \alpha; n \in I$

Solutions in the case of two equations are given

Two equations are given and we have to find the values of variable θ which may satisfy with the given equation, like

$\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$. So the common solution is $\theta = 2n\pi + \alpha,\ n \in I$. Similarly $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$. So the common solution is $\theta = 2n\pi + \alpha,\ n \in I$ and $\cos\theta = \cos\alpha$ and $\tan\theta = \tan\alpha$. So common solution is $\theta = 2n\pi + \alpha,\ n \in I$.

General solution of *a* cos θ + *b* sin θ = c, where *a*, *b*, c ∈*R* and |c| ≤ $\sqrt{a^2 + b^2}$

If $a\cos\theta + b\sin\theta = c$, Let $a = r\cos\alpha$ and $b = r\sin\alpha$, where $r = \sqrt{a^2 + b^2}$

Then, $r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = c \Rightarrow \cos(\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}} = \cos\beta$ (say) ....(1)

$\Rightarrow \theta - \alpha = 2n\pi \pm \beta = 2n\pi \pm \beta + \alpha$, where $\tan\alpha = \frac{b}{a}$, is the general solution

Alternatively, putting $a = r\sin\alpha$ and $b = r\cos\alpha$ where $r = \sqrt{a^2 + b^2} \Rightarrow \sin(\theta + \alpha) = \frac{c}{\sqrt{a^2 + b^2}} = \sin\gamma$ (say)

$\Rightarrow \theta + \alpha = n\pi + (-1)^n\gamma \Rightarrow \theta = n\pi + (-1)^n\gamma - \alpha$, where $\tan\alpha = b/a$, is the general soution.

General solution of some particular equations

(1) $\sin\theta = 0 \Rightarrow \theta = n\pi$, $\cos\theta = 0 \Rightarrow \theta = (2n+1)\pi/2$ or $n\pi + \frac{\pi}{2}$, $\tan\theta = 0 \Rightarrow \theta = n\pi$

(2) $\sin\theta = 1 \Rightarrow \theta = (4n+1)\pi/2$ or $2n\pi + \frac{\pi}{2}$, $\cos\theta = 1 \Rightarrow \theta = 2n\pi$, $\tan\theta = 1 \Rightarrow \theta = (4n+1)\pi/4$ or $n\pi + \frac{\pi}{4}$

(3) $\sin\theta = -1 \Rightarrow \theta = (4n+3)\pi/2$ or $2n\pi + \frac{3\pi}{2}$, $\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$, $\tan\theta = -1 \Rightarrow \theta = (4n-1)\pi/4$ or $n\pi - \frac{\pi}{4}$

(4) $\tan\theta$ = not defined $\Rightarrow \theta = (2n+1)\pi/2$, $\cot\theta$ = not defined $\Rightarrow \theta = n\pi$

$\operatorname{cosec}\theta$ = not defined $\Rightarrow \theta = n\pi$, $\sec\theta$ = not defined $\Rightarrow \theta = (2n+1)\pi/2$.

Method for finding principal value

Suppose we have to find the principal value of θ satisfying the equation $\sin\theta = -\frac{1}{2}$.

Since $\sin\theta$ is negative, θ will be in 3rd or 4th quadrant. We can approach 3rd or 4th quadrant from two directions. If we take anticlockwise direction the numerical value of the angle will be greater than π. If we approach it in clockwise direction the angle will be numerically less than π. For principal value, we have to take numerically smallest angle. So for principal value

(1) If the angle is in 1st or 2nd quadrant we must select anticlockwise direction and if the angle is in 3rd or 4th quadrant, we must select clockwise direction.

(2) Principal value is never numerically greater than π.

(3) Principal value always lies in the first circle (i.e., in first rotation). On the above criteria, θ will be $-\frac{\pi}{6}$ or $-\frac{5\pi}{6}$. Among these two $-\frac{\pi}{6}$ has the least numerical value. Hence $-\frac{\pi}{6}$ is the principal value of θ satisfying the equation $\sin\theta = -\frac{1}{2}$.

From the above discussion, the method for finding principal value can be summed up as follows –

(i) First draw a trigonometrical circle and mark the quadrant, in which the angle may lie.

(ii) Select anticlockwise direction for 1st and 2nd quadrants and select clockwise direction for 3rd and 4th quadrants.

(iii) Find the angle in the first rotation.

(iv) Select the numerically least angle these two values. The angle thus found will be principal value.

(v) In case, two angles one with positive sign and the other with negative sign qualify for the numerically least angle, then it is the convention to select the angle with positive sign as principal value.

Periodic function

Any function $f(x)$ is called periodic if $f(x) = f(x+k) = f(x+2k) = ...$etc, where k is called the period.

All trigonometric functions are periodic. The period of trigonometric function $\sin x, \cos x, \sec x$ and $\text{cosec } x$ is 2π because $\sin(x+2\pi) = \sin x$, $\cos(x+2\pi) = \cos x$ etc.

The period of $\tan x$ and $\cot x$ is π because $\tan(x+\pi) = \tan x$ and $\cot(x+\pi) = \cot x$

The period of the function which are of the type: $\sin ax, \cos(ax+b)$; $b\cos ax$ is $\frac{2\pi}{|a|}$

The period of $\tan ax$ and $\cot ax$ is $\frac{\pi}{|a|}$. Here $|a|$ is taken so as the value of the period is positive real number.

Some functions with their periods

Function	Period
$\sin(ax+b), \cos(ax+b), \sec(ax+b), \text{cosec } (ax+b)$	$2\pi/a$
$\tan(ax+b), \cot(ax+b)$	π/a
$\lvert\sin(ax+b)\rvert, \lvert\cos(ax+b)\rvert, \lvert\sec(ax+b)\rvert, \lvert\text{cosec}(ax+b)\rvert$	π/a
$\lvert\tan(ax+b)\rvert, \lvert\cot(ax+b)\rvert$	π/a