GEOMETRY

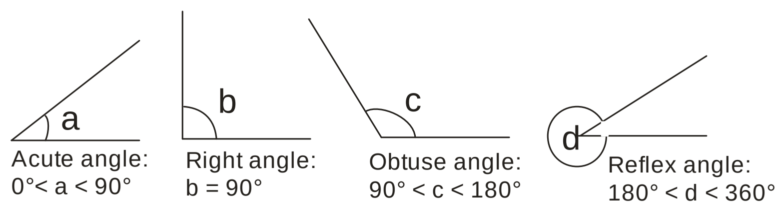
Contents

3.GEOMETRY

Geometry is concerned with the study and measurement of different shapes. The most important thing in Geometry is to get very well acquainted with different Geometrical figures and their properties.

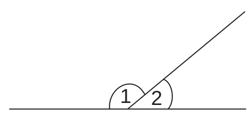
3.1 Lines and Angles

Any two straight lines which meet at a point make an angle. The angle made by the two straight lines could be any of the following types.

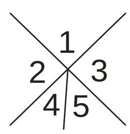


3.1.1 Basic properties of lines and angles

- Any straight line makes an angle of 180°. A pair of adjacent angles (such as ∠1 and ∠2 in the figure) that add up to 180° is called a Linear pair of angles.



- The sum of the angles made at a point is equal to 360°. So, (∠1 + ∠2 + ∠3 + ∠4 + ∠5) = 360°.

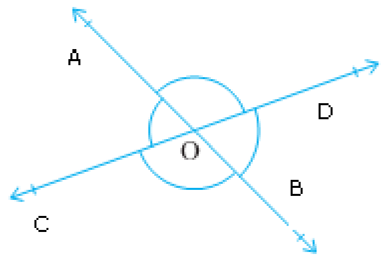


- When two lines intersect, the pair of vertically opposite angles formed is equal. So, ∠2 = ∠3 and ∠1 = ∠4+ ∠5 in the above figure.

- If the sum of two angles is 90°, they are complementary to each other; and If they add up to 180°, they are called supplementary.

- If more than two straight lines intersect at one and the same point, they are called concurrent lines.

Example: Lines AB and CD intersect each other at point O. If $\angle AOC : \angle BOC = 5 : 7$, find. $\angle AOD$ and $\angle BOD$.



Solution: $\angle AOC + \angle BOC = 180^\circ$

But $\angle AOC : \angle BOC$ = 5 : 7 (Given)

$\therefore \angle AOC = \frac{5}{12} \times 180^\circ = 75^\circ$

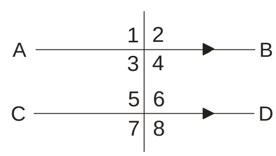
Similarly, $\angle BOC = \frac{7}{12} \times 180^\circ = 105^\circ$

$\angle AOD = \angle BOC = 105^\circ$° (Vertically opposite angles)

$\angle BOD = \angle AOC = 75^\circ$° (Vertically opposite angles)

3.1.2 Parallel Lines

Parallel lines are lines that are separated by a constant distance at all the points. They never intersect even if extended infinitely. Any line that cuts across a pair of parallel lines is called a transversal.



Properties of angles formed by the transversal with the parallel lines:

(a) Corresponding angles are equal, ∴∠1=∠5, ∠2=∠6, ∠3=∠7, ∠4= ∠8.

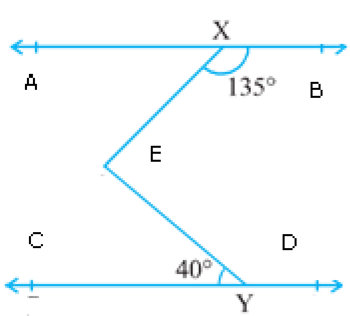
(b) Alternate interior angles are equal, ∴ ∠4 = ∠5, ∠3 = ∠6.

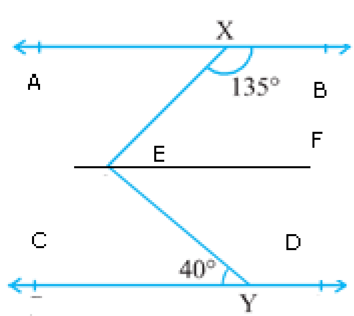
(c) The interior angles on the same side of a transversal add up to 180°, here ∠4 + ∠6 = ∠3 + ∠5 = 180°.

(d) Conversely, whenever the corresponding angles are equal or the alternate angles are equal or the interior angles are supplementary, then the two lines are parallel.

(e) Lines which are parallel to the same line are parallel to each other.

Example: If AB || CD, ∠BXE = 135° and ∠EYC = 40°, find ∠XEY.





Solution: Here, we draw a line through E and parallel to the lines AB and CD. Now, AB || EF || CD.

Now, $\angle BXE + \angle XEF = 180°$ (Interior angles on the same side of transversal XE)

So, $135° + \angle XEF = 180°$

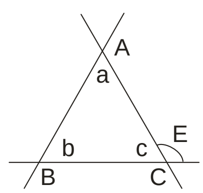
$\angle XEF = 45°$

Now, $\angle FEY = \angle EYC$ (Alternate angles)

$\therefore$ $\angle FEY = 40°$

$\angle XEF + \angle FEY = 45° + 40°$, $\therefore \angle XEY = 85°$

3.2 Triangles

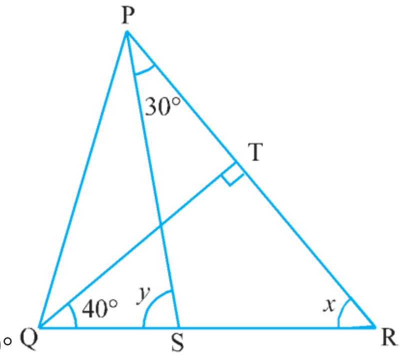


Based on angles, triangles can be acute (angled) triangle, obtuse triangle and right triangle. Based on length of sides, the different types are: scalene (none of sides are equal), isosceles (two of the sides are equal) and equilateral triangle (all 3 sides are equal).

3.2.1 Basic properties of a triangle

- Sum of the three interior angles is 180°.
- When one side is extended in any direction, the angle formed with another side is called an exterior angle (e.g. $\angle E$ in above figure).
- An exterior angle = Sum of the interior opposite angles. So, $\angle E = a + b$.
- Sum of lengths of any two sides is greater than the third side.
- The difference of any two sides is always less than the third side.
- Side opposite to the greatest angle will be the longest and vice versa.
- The area of a triangle = $(1/2) \times \text{base} \times$ corresponding height.
- If a, b, c denote the sides of a triangle then
  (i) If $c^2 < a^2 + b^2$, triangle is acute angled.
  (ii) If $c^2 = a^2 + b^2$, triangle is right angled.
  (iii) If $c^2 > a^2 + b^2$, triangle is obtuse angled.
- The ratio of areas of two triangles is equal to the ratio of the products of base and corresponding heights.
- Areas of two triangles having the same base and lying between the same parallel lines are equal.

Example: In the given figure, if $QT \perp PR$, $\angle TQR = 40°$and $\angle SPR = 30°$, find x and y.



Solution: In Δ TQR, 90° + 40° + x = 180°

(Angle sum property of a triangle)

∴ x = 50°

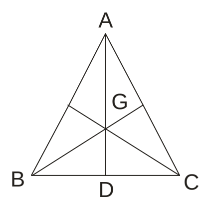
Now, y = ∠ SPR+ x (Exterior angle= sum of inter. Opp. angles)

∴ y = 30° + 50°

= 80°

3.2.2 Interior Points in a Triangle

a. Centroid

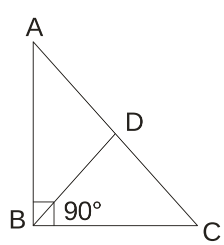


A median is a line joining a vertex to the mid point of the opposite side. The 3 medians of a triangle are concurrent and meet at centroid (G in the figure).

Properties of centroid:

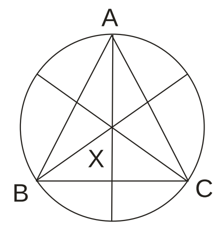
- The centroid divides the median AD in the ratio $AG:GD = 2:1$
- Any median bisects the area of the triangle. ∴Area ΔABD=Area Δ ADC.
- For a Right triangle, the Median to the hypotenuse = ½ × hypotenuse

Example: In ΔABC, $\angle B$ is a right angle, AC = 6 cm, and D is the mid-point of AC. What is the length of BD?



Solution: In a right triangle, $\mathrm{BD} = \text{half of}\,\mathrm{AC}$.∴BD = 3 cm.

b. Circumcentre

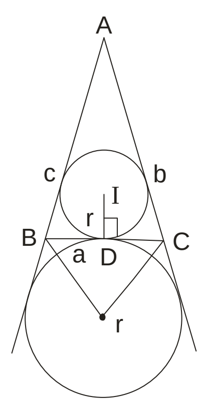


- Perpendicular bisector to any side is the line that is perpendicular to that side and passes through it's midpoint. Any point on the perpendicular bisector of a line is equidistant from the endpoints of the line.
- The perpendicular bisectors of a triangle are concurrent and meet at a point called the circumcentre. The circumcentre of a triangle is equidistant from it's three vertices. XB=XA=XC=Circumradius.
- In this triangle: $\angle BXC = 2\angle BAC, \angle AXB = 2\angle ACB, \angle CXA = 2\angle ABC$

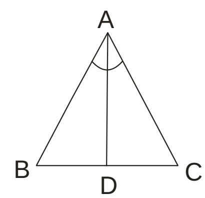
Example: A pole stands vertically inside a triangular park ABC. If the angle of elevation of the top of the pole from each corner of the park is same, then in triangle ABC, the foot of the pole is at which of the following points?
(a) Circumcentre (b) Centroid (c) Incentre (d) Orthocentre

Solution: Since angle of elevation is the same from each vertex, $\therefore$each vertex must be equi-distant from the foot of the pole. Hence the foot of the pole must lie at the circumcentre of the triangle.

c. Incentre



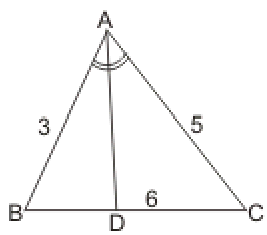
- An angle bisector is a line that divides the angle into two equal parts. Any point on the angle bisector is equidistant from the arms of the angle.
- The angle bisectors of the three angles are concurrent and meet at the incentre (I). If we take the perpendicular distance from the incentre to any side as the radius and draw a circle, it touches all the sides of the triangle. Such a circle is called the incircle and it's radius is called inradius. All the three sides are tangent to the incircle.
- The incentre divides the bisector of $\angle A$ in the ratio (b + c):a.
- Angle bisector theorem:



In the figure if AD is the angle bisector (interior), then AB/AC = BD/DC and AB × AC – BD × DC = AD^2

Example: In a triangle ABC, the lengths of the sides AB, AC and BC are 3, 5 and 6 cm respectively. If a point D on BC is drawn such that the line AD bisects the angle A internally, then what is the length of BD?

Solution: $\frac{BD}{AB} = \frac{DC}{AC}$

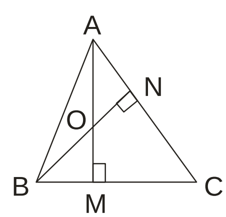


∴ $\frac{BD}{DC} = \frac{3}{5}$

∴ BD:DC = 3 : 5

Now BD = $\frac{3}{(3+5)} \times 6 = \frac{18}{8} = \frac{9}{4} = 2.25\ cm.$

d. Orthocentre



- The perpendicular drawn from vertices to the opposite sides are called the altitudes. The three altitudes meet at a common point called the orthocentre.
- Orthocentre may lie either inside or outside the triangle. In case of an obtuse triangle, it lies outside the triangle.
- The angle made by any side at the orthocenter = 180 – the opposite angle to the side. For example, in the above figure:

$$\angle BOC = 180^\circ - \angle A, \angle AOB = 180^\circ - \angle C, \angle AOC = 180^\circ - \angle B$$

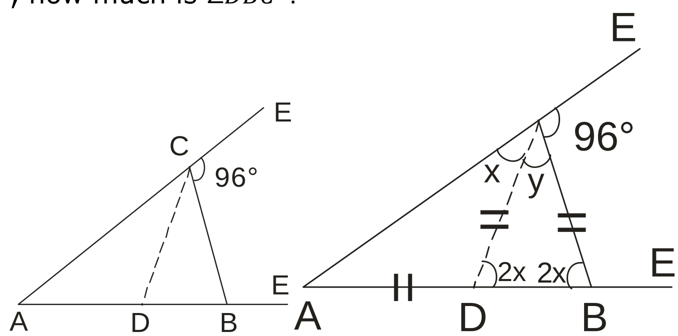
3.2.3 Special types of triangles

- **Isosceles triangle**

  If ABC is an isosceles triangle with AB = AC, the angles opposite to the equal sides, $\therefore \angle B = \angle C$. The median drawn from A to BC is also the perpendicular bisector of BC.

In Δ ABC, all the four interior points lie on the altitude drawn from the vertex A to base BC. The lengths of altitudes to the equal sides are also equal.

Example: In the figure (not drawn to scale) given below, if AD = CD = BC, and $\angle BCE = 96^{\circ}$, how much is $\angle DBC$?



Solution: Refer the diagram, $x + y = 180 - 96 = 84^{\circ}$ $(since\ AE\ is\ a\ straight\ line)$

Also in ΔCDB; $4x + y = 180$ $(angle\ sum\ property)$

Solving the above equations, $x = 32$

$\therefore$ $\angle DBC = 2x = 64^{\circ}$.

- **Equilateral Triangle**

All sides are equal and all the angles are equal to 60°.All the four interior points coincide. The medians, angle bisectors, altitudes, perpendicular bisectors of sides are all represented by same straight lines.

For a given perimeter, an equilateral triangle has the maximum area as compared to any other type of triangle.

If the side of an equilateral triangle is "a", then

It's Area = $\frac{\sqrt{3}}{4}a^2$

Altitude to any side = $\frac{\sqrt{3}}{2}a$

Example: If $a, b\ and\ c$ are the sides of a triangle, & $a^2 + b^2 + c^2 = bc + ca + ab,$ then what type of triangle are we talking about?

Solution: $Given,\ a^2 + b^2 + c^2 = bc + ca + ab$

$\therefore a^2 + b^2 + c^2 - bc - ca - ab = 0$

$\therefore 2a^2 + 2b^2 + 2c^2 - 2bc - 2ca - 2ab = 0$

$\therefore (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

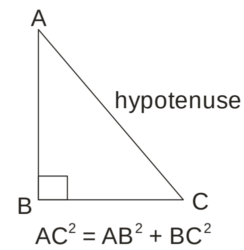
$\Rightarrow a = b = c$

$\therefore$Triangle is equilateral.

- **Right Angled Triangle**

If ABC is a right angled triangle, right angled at B, then AC is called the hypotenuse.

Pythagoras Theorem:



The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. Pythagorean triplets are sets of three integers which can be three sides of a right angled triangle. Examples of Pythagorean triplets are (3, 4, 5), (5, 12, 13) and so on.

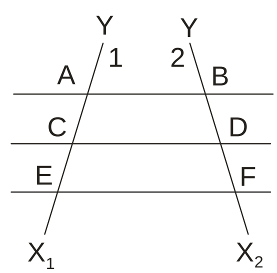
Based on Pythagoras theorem, two special types of right triangles can be defined:
$45^\circ - 45^\circ - 90^\circ Right\ triangle$: If $\angle A = \angle C = 45^\circ$, then the ratio of sides $AB:BC:AC = 1:1:\sqrt{2}$
$30^\circ - 60^\circ - 90^\circ Right\ triangle$: If $\angle A = 60^\circ \angle C = 30^\circ$, then the ratio of sides $AB:BC:AC = 1:\sqrt{3}:2$

Example: If the sides of a right triangle are: $x, x+1\ and\ x-1,$ then the hypotenuse is?

Solution: Using Pythagoras Theorem, $(x+1)^2 = x^2 + (x-1)^2$
$x^2 + 2x + 1 = x^2 + x^2 - 2x + 1$
$x^2 - 4x = 0 \quad \Rightarrow\ x = 0\ or\ 4\ \ \Rightarrow x = 4\ (x\ cannot\ be\ 0)$
$\therefore$ hypotenuse $= x + 1 = 5$

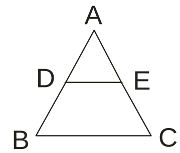
3.2.4 General Theorems on Proportionality

- **Proportionality Theorem:**



Intercepts made by two transversal lines (cutting lines) on three or more parallel lines are proportional. In the figure, lines, X_1Y_1 and X_2Y_2 are transversals to three parallel lines AB, CD, and EF. AC, CE, BD, DF are intercepts. By proportionality theorem: AC/CE = BD/DF

- **Basic Proportionality Theorem:**



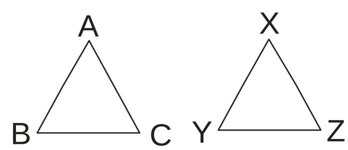
Any line drawn parallel to one side of a triangle divides the other two sides proportionally. So, if DE is drawn parallel to BC, it would divide sides AB and AC proportionally. In this case

AD/BD = AE/EC, AB/AD = AC/AE, AD/DE = AB/BC............and so on.

A special case of above theorem is: The segment joining the midpoint of any two sides of a triangle is parallel to the third side and is half the length of the third side.

3.2.5 Congruence and Similarity of Triangles

(i) Two triangles are said to be **CONGRUENT** if they are equal in all respects.



(a) The three sides of one triangle are equal to the three corresponding sides of the other.

(b) The three angles of the first are equal to the three corresponding angles of the other.

Thus, if Δ ABC and Δ XYZ are congruent, (represented as Δ ABC $\cong$ Δ XYZ), then

AB = XY, AC = XZ, BC = YZ and $\angle A = \angle X,\ \angle B = \angle Y,\ \angle C = \angle Z$.

(c) SAS criteria (Side-Angle-Side)

Δ ABC $\cong$ Δ XYZ if ***AB=XY***, $\angle A = \angle X,\ AC = XZ$

(d) AAS (Angle-Angle-Side)

ΔABC $\cong$ ΔXYZ if $\angle B = \angle Y,\ \angle C = \angle Z,\ AC = XZ$

(e) ASA (Angle-Side-Angle)

Δ ABC $\cong$ Δ XYZ if $\angle B = \angle Y,\ \angle C = \angle Z,\ BC = YZ$

(f) SSS (Side-Side-Side)

Δ ABC $\cong$ Δ XYZ if AB = XY, AC = XZ, BC = YZ

(g) RHS (Right angle-Hypotenuse-Side)

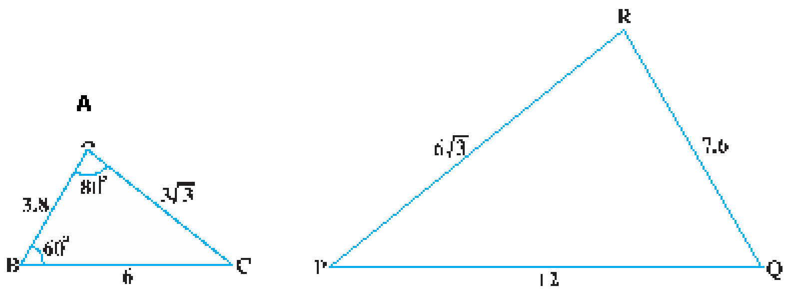
Right Δ ABC $\cong$ Right Δ XYZ if $\angle B = \angle Y$, AB = XY, AC = XZ

(ii) Two triangles are **SIMILAR** if

(a) The angles of one are respectively equal to the angles of the other (AAA criteria).

(b) The corresponding sides are proportional (SSS criteria)
(c) Similarly we have the SAS criteria.

***Example*: F**ind ∠ P.



***Solution*:** In Δ ABC and Δ PQR, $\frac{AB}{RQ} = \frac{3.8}{7.6} = \frac{1}{2}$

Similarly, $\frac{BC}{QP} = \frac{CA}{PR} = \frac{1}{2}$

$\therefore \frac{AB}{RQ} = \frac{BC}{QP} = \frac{CA}{PR}$

So, Δ ABC ~ Δ RQP (SSS similarity)

∴ ∠ C = ∠ P (Corresponding angles of similar

But ∠ C = 180° − ∠ A − ∠ B = 40° (Angle sum property)

$\therefore \angle P = 40^{\circ}$

3.2.6 Properties of Similar Triangles

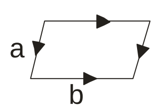
(a) If two triangles are similar, then for the proportional/ corresponding sides we find that Ratio of sides = Ratio of heights = Ratio of medians = Ratio of angle bisectors = Ratio of inradii = Ratio of circumradii
(b) Ratio of areas = Ratio of squares of corresponding sides.

3.3 Quadrilaterals

Quadrilaterals are figures enclosed by four straight lines. The important properties of the quadrilaterals are:

1. Sum of the four interior angles = 360°
2. If a quadrilateral is circumscribed about a circle, the sums of opposite sides are always equal.

3.3.1 Parallelogram



Parallelograms are a specific type of quadrilateral with following properties:

1. The opposite sides are parallel and equal.
2. Opposite angles are equal.
3. The diagonals bisect each other. (The diagonals need not to equal in length and do not necessarily bisect at right angles)
4. Sum of any two adjacent angles = 180°.
5. Each diagonal divides it into two congruent triangles.
6. A parallelogram that is inscribed in a circle is a rectangle.
7. When circumscribed about a circle, it becomes a rhombus.
8. Diagonals need not bisect angles at the vertices.
9. The figure formed by joining the mid points of the adjacent sides of any quadrilateral is a parallelogram.
10. The area of a parallelogram is (1/2) × (any side) × (height of the parallelogram on that base)
11. Parallelograms that lie on the same base and between the same pair of parallel lines have equal areas.

Example: A triangle and a parallelogram are constructed on the same base such that their areas are equal. If the altitude of the parallelogram is 100m, then the altitude of the triangle is?

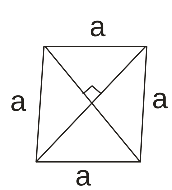
Solution: Let the common base be x m.

Since $area\ of\ triangle = area\ of\ parallelogram$

$\therefore \frac{1}{2} \times x \times altitude = x \times 100$

$\therefore altitude = 200\ m.$

3.3.2 Rhombus

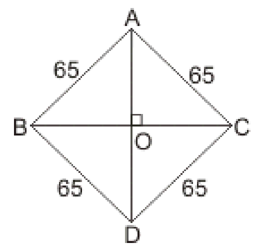


If all sides of a parallelogram are equal, it is a rhombus.

1. Opposite angles are equal
2. Diagonals bisect each other at 90°. They are not equal (except in the case of a square). They bisect the vertex angles.
3. Sum of any two adjacent angles = 180°.
4. Figure formed by joining the mid points of the adjacent sides of a rhombus is a rectangle.
5. The area of a rhombus is equal to half the product of its diagonals.

Example: The area of a rhombus is 2016 cm^2 and its side is 65 cm. What are the lengths of the diagonals (in cm)?

Solution: Let AO = x and OC = y



$\therefore x^2 + y^2 = (65)^2 = 4225$ ... (i)

Also area of rhombus, $2016 = \frac{1}{2} \times 2x \times 2y$

$\Rightarrow xy = 1008$ ...(ii)

$(x - y)^2 = x^2 + y^2 - 2xy = 4225 - 2016 = 2209$

$\Rightarrow (x - y) = 47$ ...(iii)

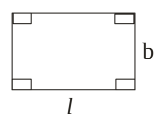
$(x + y)^2 = x^2 + y^2 + 2xy = 4225 + 2016 = 6241$

$\Rightarrow (x + y) = 79$ ...(iv)

From Equations (iii) and (iv), $x = 63$, $y = 16$

$\therefore$Lengths of diagonals are 126 cm and 32 cm.

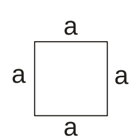
3.3.3 Rectangle



Rectangle is a parallelogram which has the following properties.

1. Opposite sides are equal and parallel. Every angle is a right angle.
2. Diagonals are equal in length and bisect each other (not at 90°).
3. Of all the rectangles of a given perimeter, a square has maximum area.
4. When a rectangle is inscribed in a circle, the diameter of the circle is equal to the diagonal of the rectangle.
5. The area of the rectangle = L $\times$ B (L is the length and B the breadth.)

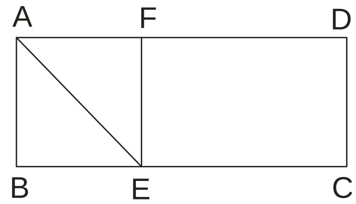
3.3.4 Square



A square is a rectangle all of whose sides are equal. It is also a rhombus all of whose angles are right angles. The other properties of a square are:

1. Diagonals bisect each other at 90° and are equal in length.
2. When inscribed in a circle, diagonal = diameter of circle.
3. When circumscribed about a circle, side of square = diameter of circle.
4. The area of a square = side^2.
5. The diagonal of a square = $\sqrt{2}$ (side).

Example: In the figure given below, $ABCD$ is a rectangle. The area of the isosceles right triangle $ABE = 7$ cm^2.; EC = 3 (BE). What is the area of $ABCD$ (in cm^2) ?



Solution: Area of ΔABE = 7 cm^2 $\Rightarrow$ $\frac{1}{2} \times AB \times BE = 7$ cm^2

$\Rightarrow$ AB^2 = 14 (ΔABE *is isosceles*)

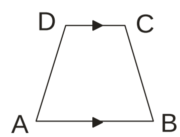
$\Rightarrow$ $AB = \sqrt{14}$ cm $= BE$

$\therefore\ EC = 3\ BE = 3\sqrt{14}$

$\therefore\ BC = 3\sqrt{14} + \sqrt{14} = 4\sqrt{14}$

$\therefore$ area rectangule $ABCD = 4\sqrt{14} \times \sqrt{14} = 56$ cm^2

3.3.5 Trapezium



A trapezium is a quadrilateral with one pair of opposite sides parallel to each other.

1. The line joining the mid-points of the oblique (non parallel) sides is half the sum of parallel sides. It is called the median of trapezium.
2. If inscribed in a circle, it becomes an isosceles trapezium. The oblique sides are equal, angles made by each parallel side with oblique sides are equal. Diagonals are equal.

3.3.6 Regular Polygon

A polygon which has all it's sides and angles equal is called a regular polygon. The segment joining any two non-adjacent vertices is called a diagonal. Polygons are named as per the number of sides they have:

A polygon with :	3 sides	—	Triangle
	4 sides	—	Quadrilateral
	5 sides	—	Pentagon
	6 sides	—	Hexagon
	7 sides	—	Septagon
	8 sides	—	Octagon
	9 sides	—	Nonagon
	10 sides	—	Decagon

1. Sum of all interior angles of a polygon is (n – 2)180°, where n is the number of sides. Hence each angle of a regular polygon is [(n – 2)180°]/n.
2. Sum of an interior angle and its adjacent exterior angle is 180°.
3. Sum of all exterior angles of a polygon is always 360°.
4. In a regular pentagon, each interior angle is 108° and each exterior angle is 72°.
5. In a regular Hexagon, each interior angle is 120° and each exterior angle is 60°.

Example: How many sides a regular polygon has with it's interior angle eight times it's exterior angle?

Solution: Let n be the number of sides
Let exterior angle be e $\therefore$Interior angle=$8e$

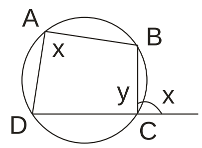
$\therefore e + 8e = 9e = 180^{\circ}$ $\Rightarrow e = 20^{\circ}$

$\therefore$ $interior\ angle = 160^{\circ}$

$\Rightarrow \frac{(n-2)\times 180^{\circ}}{n} = 160^{\circ}$

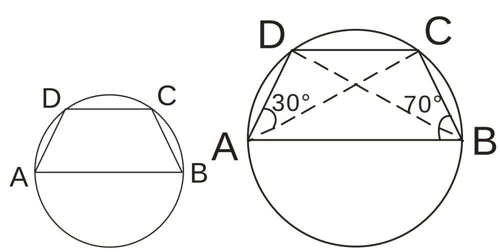
$\Rightarrow 20n = 360^{\circ}$ $\therefore n = 18$

3.3.7 Cyclic Quadrilateral



1. The four vertices lie on a circle.
2. Opposite angles are supplementary to each other.
3. If any one side is extended, exterior angle = interior opposite angle.
4. If one pair of opposite sides are equal, diagonals are equal.

Example: In the given figure, AB is diameter of the circle and points C and D are are on the circumference such that $\angle CAD = 30^{\circ}$ and $\angle CBA = 70^{\circ}$ What is the measure of $\angle ACD$?



Solution: Join DB and AC
Now $\angle DAC = \angle DBC$ ($\because$ a chord subtends equal angles on it's circumference)

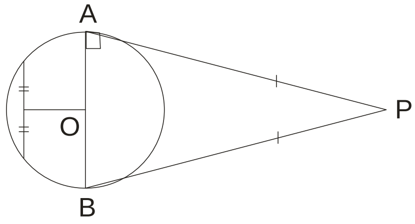
$\therefore$ $\angle CBD = 30^{\circ}$

$\Rightarrow$ $\angle DBA = 40^{\circ}$

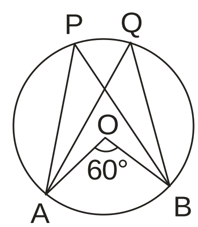
Now $\angle DBA = \angle ACD = 40^{\circ}$ (same reason as above)

3.4 Circles: properties of chords, tangents and secants

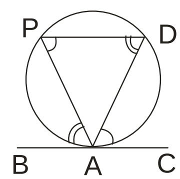
1. Tangent is perpendicular to radius at the point of contact.
2. Perpendicular from centre to a chord bisects the chord.
3. The two tangents that can be drawn form an external point are equal.



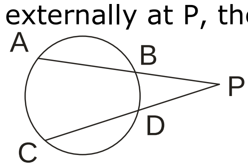
4. Equal chords of a circle are equidistant from the centre and subtend equal angles at the centre.
5. When two circles touch, their centres and the point of contact are collinear.
6. If two circles touch externally, distance between centres = sum of radii.
7. If two circles touch internally, distance between centres = difference of radii
8. Circles with same centre are concentric circles.
9. Only one circle can pass through any three given points.
10. Angle at the centre subtended by an arc = twice the angle subtended by the arc at any point on the remaining part of the circumference.
    We have $\angle$ APB = 1/2 $\angle$ AOB = 30° = $\angle$ AQB



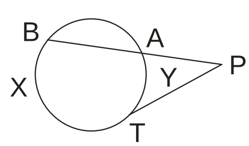
11. Angle in a semicircle is a right angle.
12. **Alternate Segment Theorem:** In the adjacent figure, if BAC is the tangent and AD is any chord, then $\angle$ DAC = $\angle$ APD or $\angle$ PAB = $\angle$ PDA **(Angles in alternate segment)**



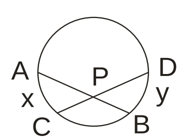
13. If two chords AB and CD intersect externally at P, then PA.PB = PC.PD



14. If PAB is a secant and PT is a tangent, then PA.PB = PT^2



15. If chords AB and CD intersect internally at P, then PA.PB = PC.PD



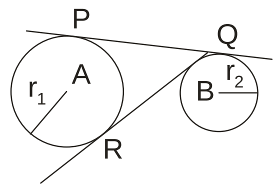
16. **Common Tangents:** For the two circles with centres A and B, PQ is a direct common tangent and RQ is a transverse common tangent.

(a) Length of direct common tangent

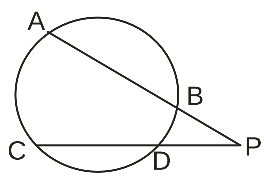
$= \sqrt{(Distance\ between\ centres)^2 - (r_1 - r_2)^2}$

(b) Length of transverse common tangent

$= \sqrt{(Distance\ between\ centres)^2 - (r_1 + r_2)^2}$



Example: If in the following figure, PA = 8 cm, PD = 4 cm, CD = 3 cm, then what is the length of AB?



Solution: We know, PC × PD = PA × PB

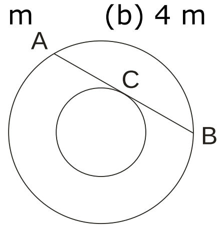
= (4 + 3) × 4 = 8 × PB

⇒ PB = $\frac{28}{8}$ = 3.5 cm

∴ AB = AP – BP = 8 – 3.5 = 4.5 cm.

Example: The line AB is 6 metres in length and is tangent to the inner one of the two concentric circles at point C. It is known that the radii of the two circles are integers. The radius of the outer circle is:

(a) 5 m (b) 4 m (c) 6 m (d) 3 m



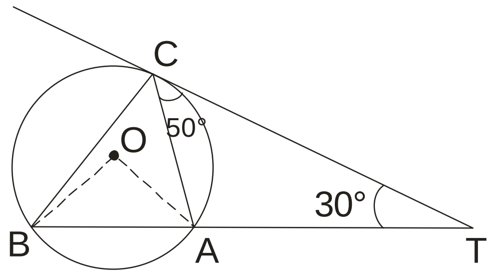
Solution: Perpendicular from the centre bisects the chord $\therefore AC = BC = 3$ m.

Out of the given options we find, if the radius of outer circle is 5 m. only then the radius of inner circle will be an integer.

$r_1^2 = (r_2)^2 + (AC)^2$ $\Rightarrow r_1^2 = (r_2)^2 + (3)^2$ $(Pythagoras\ theorem)$

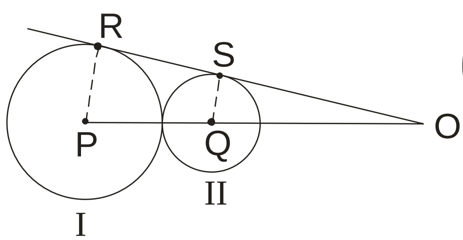
$\therefore\ r_1 = 4$ m and $r_2 = 5$ m.

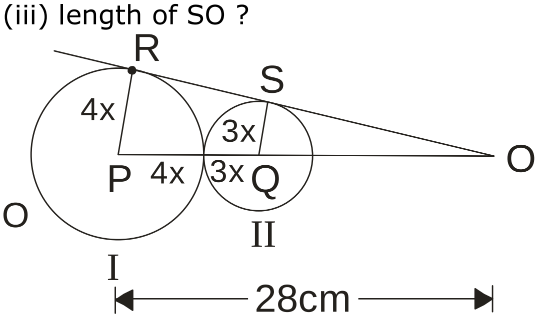
Example: In the figure given below (not drawn to scale), A, B and C are three points on a circle with centre O. The chord BA is extended to a point T such that CT becomes a tangent to the circle at point C. If $\angle ATC = 30^\circ$ and $\angle ACT = 50^\circ$, then the angle $\angle BOA$ is ?



Solution: In $\Delta\ ACT\ \angle C = 50^\circ$, $\angle T = 30^\circ\ \therefore \angle A = 100^\circ$
$\angle B = \angle ACT = 50^\circ$ $(angle\ in\ alternate\ segment)$
$\angle BAC = 50^\circ + 30^\circ = 80^\circ$ $(Exterior\ angle = sum\ of\ remote\ interior\ angles)$
$\therefore \angle BCA = 50^\circ$
$\therefore$ $\angle BOA = 2 \times \angle BCA = 100^\circ (angle\ at\ centre = twice\ angle\ on\ circum.)$

Example: In the adjoining figure, I and II are circles with centers P and Q respectively. The two circles touch each other and have a common tangent that touches them at points R and S respectively. This common tangent meets the line joining P and Q at O. The diameters of I and II are in the ratio 4:3. It is also known that the length of PO is 28 cm. What is the (i) ratio of the lengths of PQ and O ? (ii) radius of circle II ? (iii) length of SO ?





Solution: Let the radii of I and II be 4x and 3x respectively.
In ΔOSQ and ΔORP:

$\angle O = \angle O$	(common angle)
$\angle OSQ = \angle ORP = 90^\circ$	(tangent ⊥ radius)
$\therefore\ \angle OQS = \angle OPR$	(agle sum property)
$\therefore\ \Delta OSQ \sim \Delta ORP$	(AAA similarity criteria)

$\therefore\ \frac{SQ}{RP} = \frac{OQ}{OP} = \frac{3}{4}$
$\Rightarrow\ \frac{OQ}{28} = \frac{3}{4}\ \Rightarrow OQ = 21, PQ = 7$
$\therefore\ \frac{PQ}{OQ} = \frac{1}{3}$

$PQ = 4x + 3x = 7x = 7 \quad \Rightarrow x = 1$
$\therefore$ radius of circle II = $3x = 3cm$.

In ΔOSQ: $OS^2 + SQ^2 = OQ^2$
$OS^2 + 3^2 = 21^2$
$\Rightarrow OS = \sqrt{432} = 12\sqrt{3}$ cm.

More solved examples:

Example: If the areas enclosed by a circle or a square and an equilateral triangle be the same, then which of these possesses the maximum perimeter?

Solution: Let the common area be A.

For an equilateral triangle with side a, we have

$\frac{\sqrt{3}}{4}a^2 = A \text{ or } a^2 = \frac{4A}{\sqrt{3}}$ $\therefore$ Perimeter = $3 \times \sqrt{\frac{4A}{\sqrt{3}}}$ ...(i)

For a square with side 'a'

$a^2 = A$, $\therefore$ Perimeter = $4\sqrt{A}$ ...(ii)

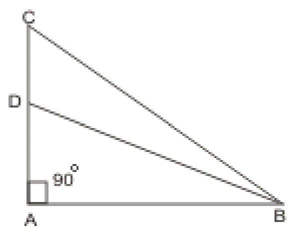
For a circle with radius 'a'

$\pi a^2 = A \Rightarrow a = \sqrt{\frac{A}{\pi}}$, $\therefore$ Perimeter = circumference= $2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\pi A}$ ...(iii)

From Equations (i), (ii) and (iii), perimeter of the equilateral triangle will be maximum.

Example: In a triangle ABC, ∠A = 90° and D is mid point of AC. The value of $BC^2 - BD^2$ is equal to how many times AD^2?

Solution: Given AC = 2AD



$BC^2 = AB^2 + AC^2$

$BD^2 = AB^2 + AD^2$

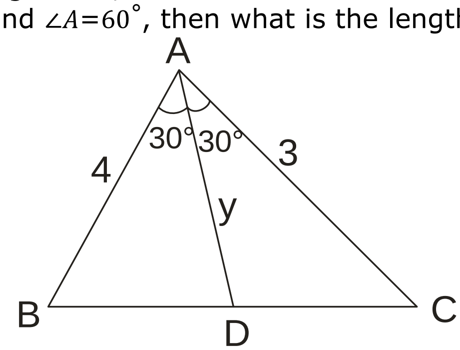
$\therefore BC^2 - BD^2$

$= AB^2 + AC^2 - AB^2 - AD^2$

$= AC^2 - AD^2 = (AC - AD)(AC + AD)$

$= (2AD - AD)(2AD + AD) = AD \times 3AD = 3AD^2$

Example: In a triangle ABC, the internal bisector of the angle A meets BC at D. If AB = 4, AC = 3 and $\angle A = 60^\circ$, then what is the length of AD?



Solution: Let BC = x and AD = y

$\frac{BD}{DC} = \frac{AB}{AC} = \frac{4}{3}$ (angle bisector theorem)

Hence, BD = $\frac{4x}{7}$ and DC = $\frac{3x}{7}$

In ΔABD, cos $30^\circ = \frac{(4)^2 + y^2 - \frac{16x^2}{49}}{2\times4\times y} \Rightarrow 4\sqrt{3}$ y= 16 + y^2 - $\frac{16x^2}{49}$ ...(i) (cosine rule)

Similarly, in ΔADC

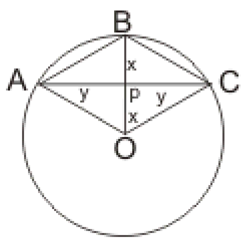
$$\cos 30^\circ = \frac{(3)^2 + y^2 - \frac{9x^2}{49}}{2\times3\times y} \Rightarrow 3\sqrt{3}\ y = 9 + y^2 - \frac{9x^2}{49} \quad \text{... (ii)}$$

From (i) and (ii), $y = \frac{12\sqrt{3}}{7}$

Example: A rhombus OABC is drawn inside a circle whose centre is at O in such a way that the vertices A, B and C of the rhombus are on the circle. If the area of the rhombus is $32\sqrt{3}$ m^2, then what is the radius of the circle?

Solution: OABC is a rhombus with centre O.

Let diagonal of the rhombus be OB = 2x and AC = 2y



Radius of the circle = OB = OA = OC = 2x

In Δ POC, $OC^2 = OP^2 + PC^2$

$(2x)^2 = x^2 + y^2$, $\Rightarrow 4x^2 = x^2 + y^2$

$\Rightarrow 3x^2 = y^2$ or $x = \frac{y}{\sqrt{3}}$ ...(i)

Also area of rhombus=$\frac{1}{2}\times(2x)(2y)$ = 32 $\sqrt{3}$

xy = $16\sqrt{3}$

$x \times x\sqrt{3} = 16\sqrt{3}$, $\Rightarrow$ x = 4 ...using (i)

∴ Radius of circle = 2 × 4 = 8 m