Definite Integral & Area Under Curves

Definition

If $\frac{d}{dx}[f(x)] = \phi(x)$ and a and b are two values independent of variable x, then $\int_a^b \phi(x)dx = [f(x)]_a^b = f(b) - f(a)$ is called definite integral of $\phi(x)$ within limits a and b. Here a is called the lower limit and b is called the upper limit of integral. The interval $[a, b]$ is known as range of integration.

(1) **Evaluation of Definite integral :** To evaluate the definite integral $\int_a^b f(x)dx$ of a continuous function $f(x)$ defined on $[a, b]$ we use following algorithm.

Algorithm :

Step I : Find the indefinite integral $\int f(x)dx$. Let this be $\phi(x)$. There is no need to keep the constant of integration.

Step II : Evaluate $\phi(b)$ and $\phi(a)$.

Step III : Calculate $\phi(b) - \phi(a)$.

The number obtained in step III is the value of the definite integral $\int_a^b f(x)dx$.

(2) **Properties of Definite integral :** (a) $\int_a^b f(x)dx = \int_a^b f(t)dt$ *i.e.*, integration is independent of the change of variable.

(b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

(d) **Generalization** The above property can be generalized into the following form

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_n}^b f(x)dx, \text{ where } a < c_1 < c_2 < c_3 ...... < c_{n-1} < c_n < b .$$

(e) $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function or } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is an odd function or } f(-x) = -f(x) \end{cases}$

(f) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$

(g) (i) $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

(ii) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

(h) If $f(x)$ is a periodic function with period T then, $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$

(i) If $f(x)$ is a periodic function with period T and $a \in R^+$, then $\int_{nT}^{a+nT} f(x)dx = \int_0^{a} f(x)dx$

(j) **Leibnitz's rule** for the differentiation under the integral sign :

If the functions $\phi(x)$ and $\psi(x)$ are defined on [a, b] and differentiable at a point $x \in (a,b)$ and $f(x,t)$ is continuous, then, $\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)dt\right) = \frac{d}{dx}\{\psi(x)\}f(\psi(x)) - \frac{d}{dx}\{\phi(x)\}f(\phi(x))$.

(k) If $f(x) \geq 0$ on the interval [a, b], then $\int_a^b f(x)dx \geq 0$.

(l) If $f(x) \leq g(x)$ on [a, b] then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

(m)If m and M are the smallest and greatest values of a function $f(x)$ defined on an interval [a, b], then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

(n) If $f(x)$ is a periodic function with period T, then $\int_a^{a+T} f(x)dx$ is independent of a.

Some deductions

(1) **Property (e) :** (i) $\int_{-a}^{a} f(x^2)dx = 2\int_0^{a} f(x^2)dx$, because $f(x^2)$ is an even function.

(ii) $\int_{-a}^{a} xf(x^2)dx = 0$, because $xf(x^2)$ is an odd function.

(2) **Property (f) :** This properties is used when the upper limit is to be half

$\int_0^{\pi} f(\sin x)dx = 2\int_0^{\pi/2} f(\sin x)dx$, because $f(\pi - x) = f(x)$

(3) **Property (g) :** (i) $\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx$ (ii) $\int_0^{\pi/2} f(\tan x)dx = \int_0^{\pi/2} f(\cot x)dx$

(iii) $\int_0^{1} f(\log x)dx = \int_0^{1} f\{\log(1-x)\}dx$ (iv) $\int_0^{a} \frac{f(x)}{f(x)+f(a-x)}dx = \frac{a}{2}$ (v) $\int_0^{\pi/2} \frac{f(x)}{f(x)+f(\pi/2-x)}dx = \frac{\pi}{4}$

(vi) $\int_0^{\pi/2} f(\sin 2x)\sin x\, dx = \int_0^{\pi/2} f(\sin 2x)\cos x\, dx$ (vii) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}dx = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x+\sin^n x}dx = \frac{\pi}{4}$

(viii) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}dx = \frac{\pi}{4}$ (ix) $\int_0^{\pi/2} \frac{1}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{1}{1+\cot^n x}dx = \frac{\pi}{4}$

(x) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x+\text{cosec}^n x}dx = \int_0^{\pi/2} \frac{\text{cosec}^n x}{\text{cosec}^n x+\sec^n x}dx = \frac{\pi}{4}$ (xi) $\int_0^{\pi/4} \log(1+\tan x)dx = (\pi/8)\log 2$

Summation of series by integration

For finding sum of an infinite series with the help of definite integral, following formula is used $\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$

The following method is used to solve the questions on summation of series.

(i) After writing $(r-1)^{th}$ or r^{th} term of the series express it in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$. Therefore the given series will take the form $\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n}$

(ii) Now writing $\int$ in place of $\left(\lim_{n\to\infty}\Sigma\right)$, x in the place of $\left(\frac{r}{n}\right)$ and dx in the place of $\frac{1}{n}$, we get the $\int f(x)dx$ in the place of above series.

(iii)The lower limit of this integral $= \lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=0}$

Where r = 0 is taken corresponding to first term of the series and upper limit, $\lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=n-1}$

Gamma function

If n is a positive rational number, then the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is defined as Gamma function and is denoted by Γn *i.e.*. $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$, where $x \in Q^+$.

(1) **Properties of Gamma function :** Gamma function has the following properties:

(i) $\Gamma 1 = 1, \Gamma 0 = \infty$ and $\Gamma(n+1) = n\Gamma n$.

For example, $\Gamma 5 = 4\Gamma 4 = 4\times 3\Gamma 3 = 4\times 3\times 2\Gamma 2 = 4\times 3\times 2\times 1\Gamma 1 = 4\times 3\times 2\times 1$.

(ii) If $n \in N$, then $\Gamma(n+1) = n!$

(iii) $\Gamma(1/2) = \sqrt{\pi}$

(iv) $\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

Walli's formulae

(1) $$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases}\dfrac{(n-1)(n-3)(n-5)........4.2}{n(n-2)(n-4).........3.1}, \text{if } n \text{ is odd}\\[2ex] \dfrac{(n-1)(n-3)(n-5).....3.1}{n(n-2)(n-4).....4.2}.\dfrac{\pi}{2}, \text{if } n \text{ is even}\end{cases}$$

(2) $\int_0^{\pi/2} \sin^m x.\cos^n x \, dx$, $m,n \in N$ $= \dfrac{(m-1)(m-3)(m-5).....2 \text{ or } 1.(n-1)(n-3)......2 \text{ or } 1}{(m+n)(m+n-2)......2 \text{ or } 1} K$

here $K = \pi/2$ if both m and n are even = 1 if in both m and n at least one is odd.

Some important results

(1) $\int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log(\cos x)dx = \dfrac{-\pi}{2}\log 2$

(2) $\int_0^{\pi/2} \log(\tan x)dx = \int_0^{\pi/2} \log(\cot x)dx = 0$

(3) $\int_\alpha^\beta \dfrac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \pi, (\beta > \alpha)$

(4) $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}dx = \dfrac{\pi}{8}(\beta-\alpha)^2, (\beta < \alpha)$

(5) $\int_0^\pi \cos mx . \cos nx \, dx = 0$, when $m \neq n$

(6) $\int_0^\pi \sin mx . \sin nx \, dx = 0$, when $m \neq n$.

Area Under Curves

Curve tracing

For solving the problem on area under curves easily, if possible, first draw the rough sketch of the required area, hence following are some steps to draw the sketch of area.

(1) **Symmetry :** The curve is symmetrical about x - axis if equation of the curve contain only even power of y. Similarly it is symmetrical about y-axis if equation of the curve contain even power of x. If in the equation of the curve, power of x and y are even, then curve is symmetric about both axis. When equation of curve remains unchanged after replacing x to $-x$ and y to $-y$, then curve is symmetric in opposite quadrant.

(2) **Origin :** If the equation of the curve contains no constant term then it passes through the origin

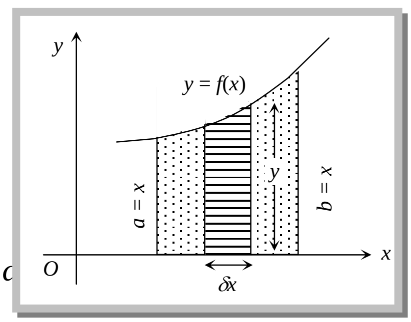
(3) **Point of intersection with the axes :** We get real values of x on putting $y = 0$ in the equation of the curve, the real values of x and $y = 0$ give those points, where the curve cut the x-axis, similarly putting $x = 0$. We get the point of intersection of the curve and y – axis.

Area bounded by curve

(1) The area bounded by a curve $y = f(x)$, x- axis and ordinates $x = a$ and $x = b$ is

$$= \int_a^b y \, dx$$

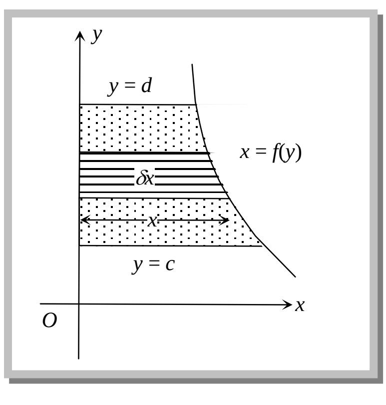
$$= \int_a^b f(x) dx$$



(2) The area bounded by a curve $x = f(y)$, y-axis and abscissa $y = c$ and $y = c$

$$= \int_c^d x \, dy$$

$$= \int_c^d f(y) dy$$



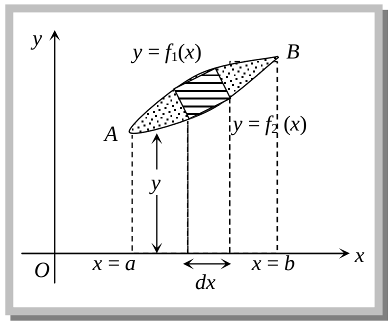
Positive and Negative area

Area is always taken as positive. If some part of area lies above the x-axis and some part lies below x-axis, then the area of two parts should be calculated separately and then add their numerical values to get the desired area.

Area between two curves

(1) When both curves intersects at two points and their common area lies between these points :

If the curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$, where $f_1(x) > f_2(x)$ intersects in two points A $(x = a)$ and $B(x=b)$, then common area between the curves is

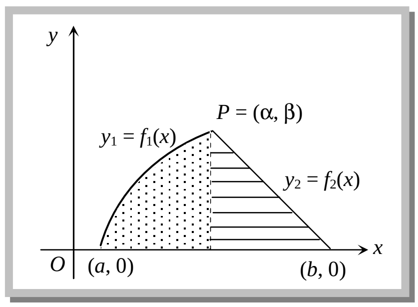


$$= \int_a^b (y_1 - y_2)dx$$

$$= \int_a^b [f_1(x) - f_2(x)]dx$$

(2) When two curves intersect at a point and the area between them is bounded by x-axis –

Area bounded by the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$ and x-axis is



$$= \int_a^\alpha f_1(x)dx + \int_\alpha^b f_2(x)dx$$

Where (α, β) is the point of intersection of two curve.