TRIPLE PRODUCT OF VECTOR

Contents

1. TRIPLE PRODUCT OF VECTOR

1.1 Introduction

The triple product is a product of three 3-dimensional vectors, usually Euclidean vectors. The name "triple product" is used for two different products, the scalar-valued scalar triple product and, less often, the vector-valued vector triple product.

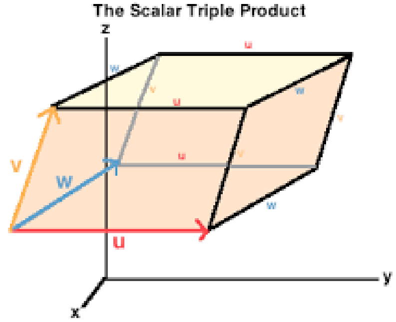
1.2 Scalar triple product

The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$(denoted as $[\vec{a}\vec{b}\vec{c}]$) is defined as $(\vec{a} \times \vec{b}).\vec{c}$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}.$, $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$. then:

$$(\vec{a} \times \vec{b}).\vec{c} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

The Scalar Triple Product



The absolute value of the scalar triple product represents the volume on the parallelepiped formed between three vectors.

Properties of Scalar triple product

(i) $(\vec{a} \times \vec{b}).\vec{c} = \vec{a}.(\vec{b} \times \vec{c})$

(ii) $[\vec{a}\vec{b}\vec{c}] = [\vec{b}\vec{c}\vec{a}] = [\vec{c}\vec{a}\vec{b}] = -[\vec{b}\vec{a}\vec{c}] = -[\vec{c}\vec{b}\vec{a}] = -[\vec{a}\vec{c}\vec{b}]$

(iii) $|(\vec{a} \times \vec{b}).\vec{c}|$ represent the volume of the parallelepiped whose adjacent sides are represented by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Therefore three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $[\vec{a}\vec{b}\vec{c}] = 0$ i.e., $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}. = 0$

(iv) Volume of the tetrahedron = $\frac{1}{6}$ | $[\vec{a}\vec{b}\vec{c}]$ |.

(v) $[\vec{a}+\vec{b}\vec{c}\vec{d}] = [\vec{a}\vec{c}\vec{d}] + [\vec{b}\vec{c}\vec{d}]$

(vi) If any two vectors among $\vec{a}, \vec{b}, \vec{c}$ are equal, then $[\vec{a}\vec{b}\vec{c}]$= 0.

(vii) Three vectors $\vec{a}, \vec{b}, \vec{c}$ form a right-handed or a left-handed system, according as $[\vec{a}\vec{b}\vec{c}]$> or< 0.

(viii) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$

(ix) $[\vec{a}\vec{b}\vec{c}][\vec{u}\vec{v}\vec{w}] = \begin{vmatrix} \vec{a}.\vec{u} & \vec{b}.\vec{u} & \vec{c}.\vec{u} \\ \vec{a}.\vec{v} & \vec{b}.\vec{v} & \vec{c}.\vec{v} \\ \vec{a}.\vec{w} & \vec{b}.\vec{w} & \vec{c}.\vec{w} \end{vmatrix}$

(x) Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ will be coplanar if $[\vec{d}\vec{b}\vec{c}] + [\vec{d}\vec{c}\vec{a}] + [\vec{d}\vec{a}\vec{b}] = [\vec{a}\vec{b}\vec{c}]$ or equivalently $[(\vec{b}-\vec{a})(\vec{c}-\vec{a})(\vec{d}-\vec{a})] = 0$

Important Notes on Scalar Triple Product

- [a, b, c] = [b, c, a] = [c, a, b]
- [a (b+c) d] = [a b d] + [a c d], [a b (c+d)] = [a b c] + [a b d]
- [λa b c] = [a λb c] = [a b λc] = λ [a b c], where λ is a real number.
- The scalar triple product of three non-zero vectors is zero if and only if they are coplanar.

Illustration: Prove that $[\vec{a}+\vec{b}\vec{b}+\vec{c}\vec{c}+\vec{a}] = 2[\vec{a}\vec{b}\vec{c}]$

Solution: $[\vec{a}+\vec{b}\vec{b}+\vec{c}\vec{c}+\vec{a}] = (\vec{a}+\vec{b}).(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})$

= $(\vec{a}+\vec{b}).(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a})$

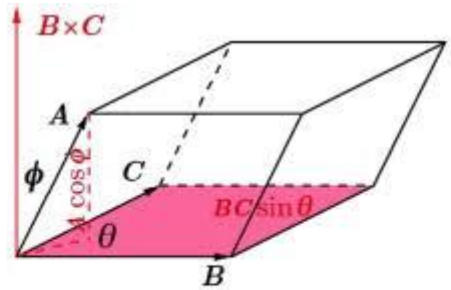
= $(\vec{a}.\vec{b}\times\vec{c}+\vec{b}.\vec{c}\times\vec{a})$

= $[\vec{a}\vec{b}\vec{c}] + [\vec{a}\vec{b}\vec{c}]$

= 2 $[\vec{a}\vec{b}\vec{c}]$

1.3 Vector triple product

The vector triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as the vector $\vec{a}\times(\vec{b}\times\vec{c})$. If at least one of $\vec{a}, \vec{b}$ and $\vec{c}$ is a zero vector or $\vec{b}$ and $\vec{c}$ are collinear vectors or $\vec{a}$is perpendicular to both $\vec{b}$ and $\vec{c}$, only then $\vec{a}\times(\vec{b}\times\vec{c})$.= $\vec{0}$. In all other cases $\vec{a}\times(\vec{b}\times\vec{c})$ will be a non-zero vector in the plane of non-collinear vector $\vec{b}$ and $\vec{c}$ and perpendicular to the vector $\vec{a}$.



Properties of Vector triple product

(I) $\vec{a} \times (\vec{b} \times \vec{c})$. $= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(ii) In general, $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(iii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ if and only if $\vec{a}$ and $\vec{c}$ are collinear.

(iv) We can express $\vec{a} \times (\vec{b} \times \vec{c}) = \lambda\vec{b} + \mu\vec{c}$, for some scalars λ and μ.

Illustration: For any vector $\vec{a}$, prove that

$$\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}.$$

Solution: $[\hat{\imath} \times (\vec{a} \times \hat{\imath})] + [\hat{\jmath} \times (\vec{a} \times \hat{\jmath})] + [\hat{k} \times (\vec{a} \times \hat{k})]$

$=[(\hat{\imath}.\hat{\imath})\vec{a} - (\hat{\imath}.\vec{a})\hat{\imath}] + [(\hat{\jmath}.\hat{\jmath})\vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath}] + [(\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}]$

$=\vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} + \vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath} + \vec{a} - (\hat{k}.\vec{a})\hat{k}$

$=3\vec{a} - [(\hat{\imath}.\vec{a})\hat{\imath} + (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\vec{a})\hat{k}]$

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$. Then

$\hat{\imath}\vec{a} = \hat{\imath}(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) = a_1\hat{\imath}^2 + a_2(\hat{\imath}.\hat{\jmath}) + a_3(\hat{\imath}.\hat{k}) = a_1(1) + 0 + 0 = a_1$

Similarly, $\hat{\jmath}.\vec{a} = a_2\hat{k}.\vec{a} = a_3$

$\therefore$ LHS $= 3\vec{a} - (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) = 3\vec{a} - \vec{a} = 2\vec{a}$= RHS.