STRAIGHT LINE AND FAMILY OF STRAIGHT LINES

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- **Slope of a line:** Slope of a line is the tangent of the angle which the line makes with the positive direction of the x-axis. It is usually denoted by m. If a line makes an angle θ with the positive direction of the x-axis. Then $m = \tan \theta$.

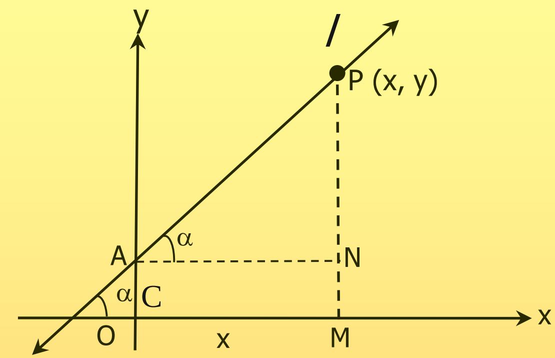
- **Intercepts:** If a line cuts the x-axis at A and y-axis at B, then OA is called intercept on x-axis and OB is called intercept on y-axis.

- **Slope-intercept form of the equation of a line:** A line l may be uniquely determined when its slope and y-intercept are given. The equation thus obtained is called slope-intercept from of the equation of the line.

y = mx + c

i.e. y = (slope of the line)x + (y-intercept)

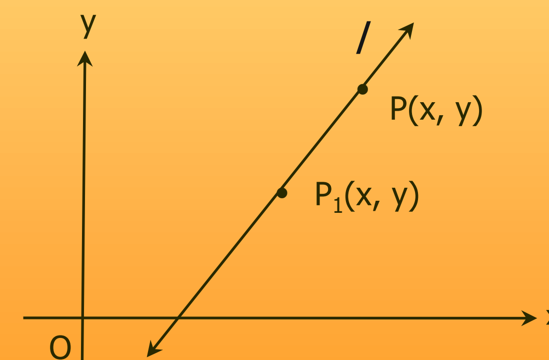
This is the equation of the line in *slope-intercept form.*



Corollary : The equation of a line with slope m and the x-intercept d is y = m (x-d).

- **Point-slope form of the equation of a line:** $m = \frac{y - y_1}{x - x_1}$

  or $y - y_1 = m(x - x_1)$



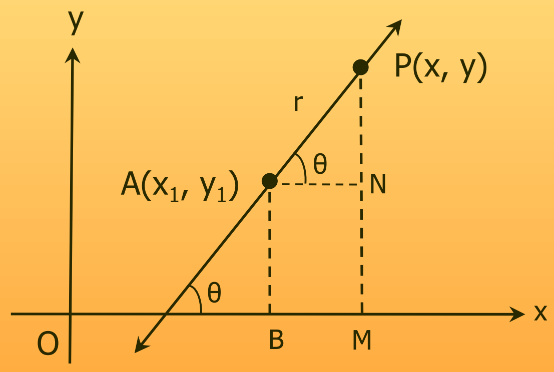
The equation $y - y_1 = m(x - x_1)$ represents the set of all those points which lie on line *l* through point (x_1, y_1) and whose slope is m.

This form of equation of a line is called *point-slope form.*

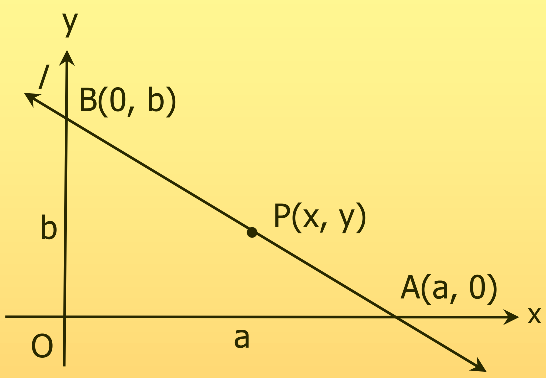
Note: If the line through $P_1(x_1, y_1)$ is parallel to y-axis, then its slope m will not be defined. Hence, the point slope form of the equation of a line is not applicable in this case.

- **Symmetric form and parametric equation of a line:** Let a line *l* pass through a point A (x_1, y_1) and be inclined at an angle θ with the positive direction of x-axis. Then, the equation involving x_1, y_1 and θ is called the *symmetric forms* of the equations of the line *l*.

  Thus, $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$ are called *parametric forms* of the equations of a line, r being the parameter. For different values of r, we can find different points of the line.



- **Intercept form of the equation of a line:** Let *a* and *b* be the intercepts made by the line on x-axis and y-axis, respectively.

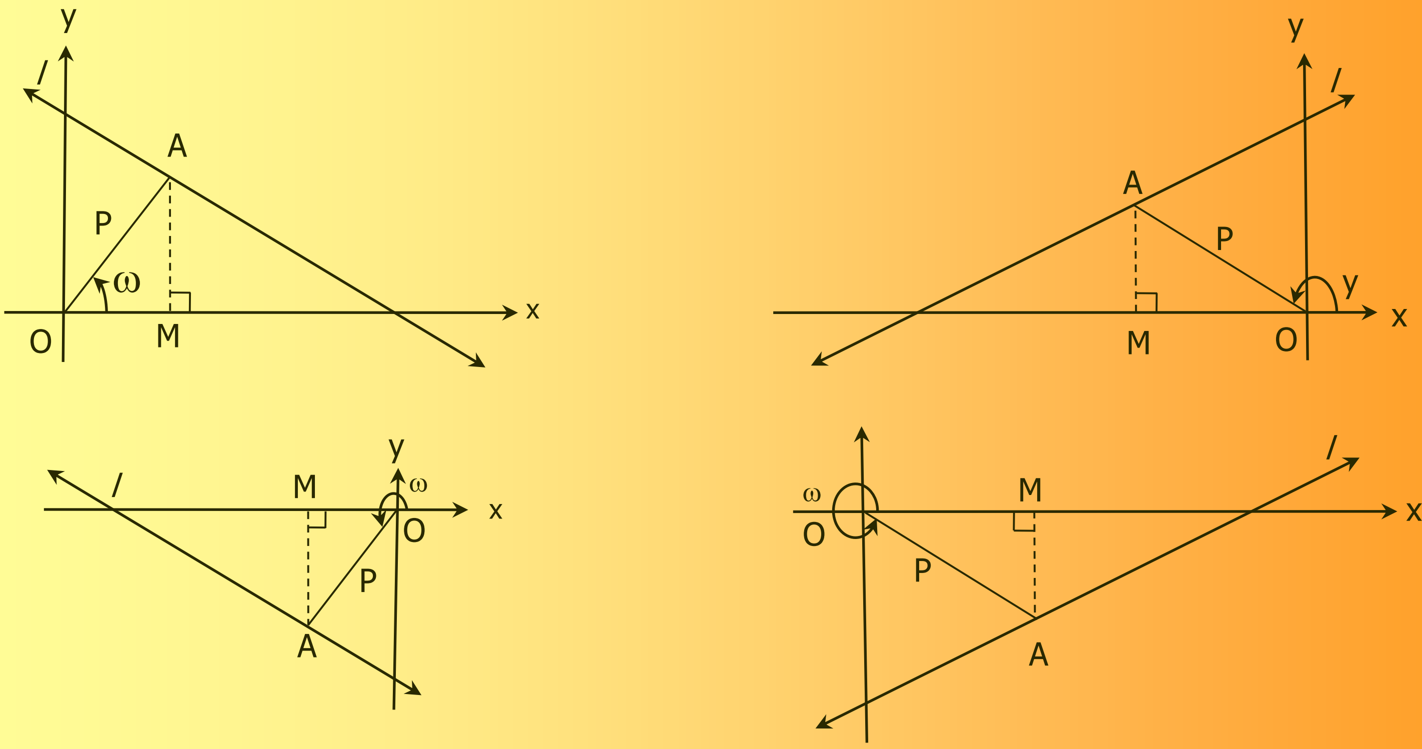


$$\frac{x}{a}+\frac{y}{b}=1$$

$$i.e., \frac{x}{\text{x-intercept}}+\frac{y}{\text{y-intercept}}=1.$$

- **Normal form of the equation of a line:** Let *l* be the given line and OA be the perpendicular drawn from origin to the line *l*.

Let OA = p and $\angle$ XOA = ω



The possible positions of the line l in the xy-plane are as shown in Fig. [(i)] to (iv)].

Draw perpendicular AM on the x-axis in each case.
We have OM = $p \cos \omega$ and MA = $p \sin \omega$.
Thus, the coordinates of the point A are ($p \cos \omega$, $p \sin \omega$).
Also, the slope of the line OA = $\tan \omega$.
Therefore, the slope of the line *l* which is perpendicular to OA is given by $m = \dfrac{-1}{\text{slope of OA}} = \dfrac{-1}{\tan \omega} = \dfrac{-\cos \omega}{\sin \omega}$

We know the slope of the line *l* and a point P ($p \cos \omega$, $p \sin \omega$) on the line.
Therefore, by point –slope form, the equation of the line *l* is

$$y - p \sin \omega = \frac{-\cos \omega}{\sin \omega}(x - p \cos \omega)$$

or $y \sin \omega - p \sin^2 \omega = -x \cos \omega + p \cos^2 \omega$
or $x \cos \omega + y \sin \omega = p (\sin^2 \omega + \cos^2 \omega)$
or $x \cos \omega + y \sin \omega = p$.
This is called the *normal form* of the equation of a line.

- **General Equation of a Straight Line :**

Theorem 1. *A general linear equation* Ax + By + C = 0, *provided* A *and* B *are not both zero simultaneously, always represents a straight line.*

Case (i): If the line is parallel to or coincident with the y-axis, then its equation can be taken as x = a (a = 0 in case the line coincides with the y-axis). This equation x-a = 0 is a linear equation in x and y with coefficient of y as zero and coefficient of x as 1.

Case (ii): If the line cuts y-axis, then it must have some slope m and intercept c on y-axis. The equation of this line can be put in the form

y = m x + c or m x – y + c = 0,

which is a linear equation in two variables x and y.

Theorem 2. *Every straight line has an equation of the form* $Ax + By + C = 0$, *where* A, B *and* C *are consonants.*

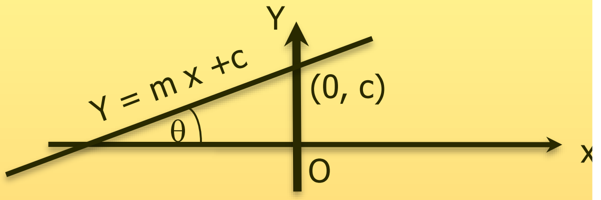
Combining theorems 1 and 2, we conclude that a general equation of a line is $Ax + By + C = 0$.

Thus, given any two conditions about the line the equation of the line can be uniquely determined, as it contains only two independent constants.

- **Equation of a line parallel to y-axis:** $x = a$ is the equation of a line parallel to y-axis and at a distance of 'a' units from y-axis.
- **Equation of y-axis:** It is given by $x = 0$
- **Equation of a line parallel to x-axis:** $y = a$ is the equation of a line parallel to x-axis and at a distance of 'a' units from x-axis.

- **Equation of x-axis:** y= 0 is the equation of x-axis.

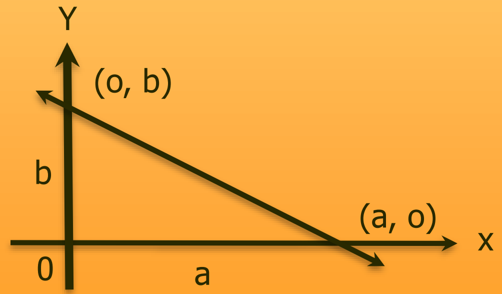
- **Explicit equation:** *(Slope form or Tangent form)*.
  Y = mx + c, where m = tan θ, θ is the angle which the straight line makes with the positive direction of x-axis and c is the intercept on y-axis.



- **Particular form:** y = mx is equation of line passing through the origin.

- **Intercept form (or Segmentary equation):**

$$\frac{x}{a}+\frac{y}{b}=1$$



where a and b are the intercepts on x and y-axis respectively.

- **One-point form:** *Straight lines through one-point*. Given a point P_1 (x_1, y_1), the equation of the straight lines passing through P_1 is
  $a(x-x_1) + b(y-y_1) = 0$
  On changing a and b, different straight lines are obtained. The straight lines (excluding that which is parallel with the y-axis) are represented also by the equation
  $y - y_1 = m(x - x_1)$
- **Two-point form:** *Straight line passing through two points*. Given the points P_1 (x_1, y_1) and P_2 (x_2, y_2), the equation of the straight line passing through P_1 and P_2 is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or by } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

- **Normal form or perpendicular form:**
  x cos α + γ sin α = *p* is the equation of a line in the normal form. *P* is the length of the perpendicular drawn from the origin on the line and α is the angle which this perpendicular makes with the positive direction of x-axis.

- **Distance form:** $\dfrac{x-x_1}{\cos\theta} = \dfrac{y-y_1}{\sin\theta} = r$

  is the equation of a line which passes through the point (x_1, y_1) and makes an angle θ with the x-axis and r is the distance of the point (x, y) from (x_1, y_1).

- **Implicit Equation:** Every equation of the first degree ax + by + c = 0 in x and y represents a straight line.

$$\text{Its slope is} = \frac{-a}{b} = \frac{\text{coeff. of x}}{\text{coeff. of y}}$$

- **Intersection between two straight lines**
- **Explicit form:** Given the two straight lines

  $y = mx + c$ ...(i)

  $y = m_1x + c_1$ ...(ii)

  the co-ordinates of any points in common between the two straight lines are obtained by solving the system of equation (i) and (ii).

  If $m \neq m_1$ the straight lines are *incident* and have in common only one single point P.

  If m = m and $c \neq c_1$, the straight lines are *parallel* and distinct and do not have any point in common (*the system does not have a solution*).

  If $m = m_1$ and $c = c_1$, the straight lines are *coincident* and, therefore, have infinite points in common (*the system has infinte solutions*).

- **Implicit form:** Given the two straight lines
  $ax + by + c = 0$ …(i)
  $a_1x + b_1y + c_1 = 0$ …(ii)
  the co-ordinates of any points in common between the two straight lines are obtained by resolving the system of equations (i) and (ii).
  Angle between two straight lines. Given the two straight lines

  $$ax + by + c = 0$$

  and $$a_1x + b_1y + c_1 = 0$$

  the angle α formed by them is given by $\tan\alpha = \dfrac{ab_1 - a_1b}{aa_1 + bb_1}$

  *Explicit form***,** Given the two straight lines: $y = mx + c$ and $y = m_1x + c_1$ and the angle α formed by them is given by

  $$\tan\alpha = \frac{m_1 - m}{1 + m_1m}$$

- **Condition of parallelism:** The two straight lines $y = mx + c$ and $y = m_1x + c_1$ are parallel if $\tan \alpha = 0$

  i.e. $m = m_1$ (Explicit form)

  and $\frac{a}{a_1} = \frac{b}{b_1}$ (Implicit form)

- **Condition of orthogonality of straight lines:** The two straight lines $y = mx + c$ and $y = m_1x + c_1$ are orthogonal if $\alpha = 90^\circ$ i.e., $\tan \alpha = \infty$

  $$\therefore m = -\frac{1}{m_1}$$

  or $m\, m_1 = -1$ (Explicit form)

  $$\frac{a}{b} = \frac{-b_1}{a_1}$$

  or $aa_1 + bb_1 = 0$ (Implicit form)

If $\frac{a}{a_1} \neq \frac{b}{b_1}$, the straight lines are incident and have common only one single point P.

If $\frac{a}{a_1} \neq \frac{b}{b_1} \neq \frac{c}{c_1}$, the straight lines are parallel and distinct and do not have any point in common (*the system does not have a solution*).

If $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$, the straight lines are coincident and, therefore, have infinite points in common (*the system has infinite solution*).

• **Angle Between Two Lines:** The angle between the lines l_1 and l_2 is either acute or obtuse, as shown in Fig (i) and (ii).

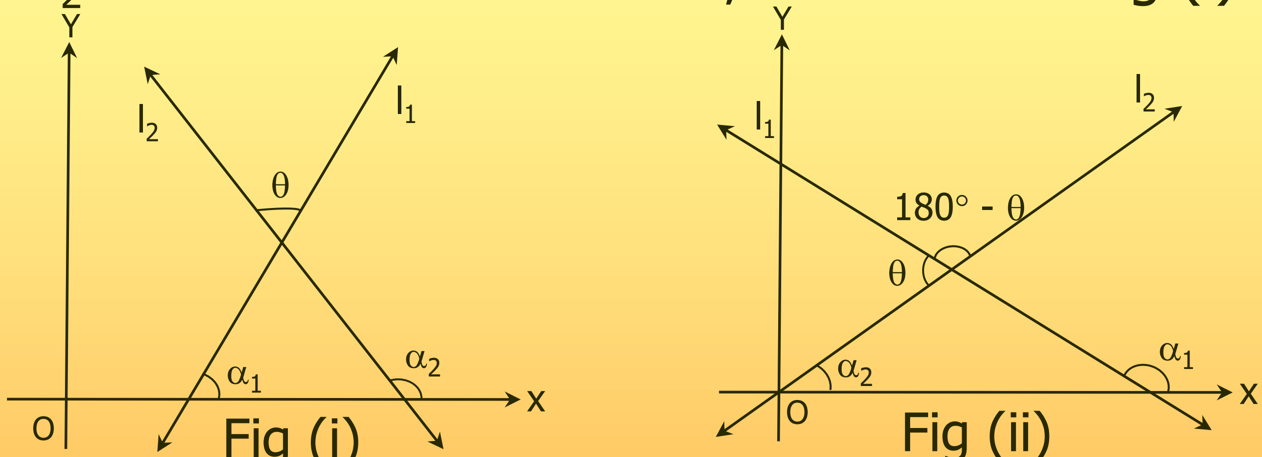


Fig (i) Fig (ii)

Let m_1 and m_2 be slopes, and α_1 and α_2 be the angles made by lines l_1 and l_2 with positive direction of the x-axis, respectively, so that

$m_1 = \tan \alpha_1$ and $m_2 = \tan \alpha_2$.

Now, in Fig. (1), we see that

$\alpha_2 = \alpha_1 + \theta$

so that $\theta = \alpha_2 - \alpha_1$ and $\tan \theta = \tan(\alpha_2 - \alpha_1)$,

$$\text{or } \tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}$$

$$\text{Therefore } \tan\theta = \frac{m_2 - m_1}{1 + m_1\, \text{m}_2}, \theta \neq 90°.$$

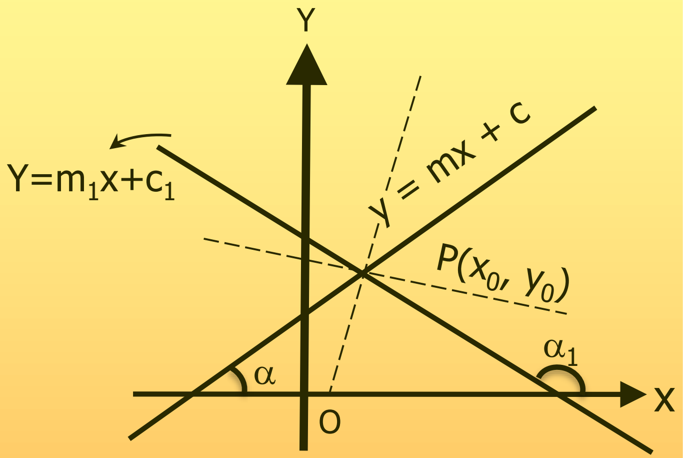
Remarks: *In numerical examples, sometimes tan θ is found to be negative. This would mean that instead of acute angle between the lines, its supplement is obtained, which is also the angle between the lines.*

- **Perpendicular Distance of a point from a straight line:** Given the point P (x_1, y_1) and the straight line ax + by + c= 0, the distance of the point P from the straight line is given by

$$\text{d} = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

- **Bisectors of the angles formed by two straight lines**

(a) *Explicit form*. Given two straight lines $y = mx + c$ *and* $y = m_1x + c_1$ which intersect at the point P (x_0, y_0), the equations of the lines bisecting the angles and consisting of two straight line are



$$y - y_0 = a(x - x_0)$$

$$y - y_0 = -\frac{1}{a}(x - x_0)$$

(b) Implicit form. Given the straight lines

$ax + by + c = 0$

and $a_1x + b_1y + c_1 = 0$.

The equation of the bisectors of the angles, consisting of two straight lines are $\frac{ax+by+c}{\sqrt{a^2+b^2}} = \pm\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$

- **Family of lines passing through the point of intersection of two given lines:** The equation of family of lines passing through the point of intersection of two given lines

  $$a_1x + b_1y + c_1 = 0$$

  and $$a_2x + b_2y + c_2 = 0$$

  is $$(a_1x + b_1y + c_1) + k\,(a_2x + b_2y + c_2) = 0$$

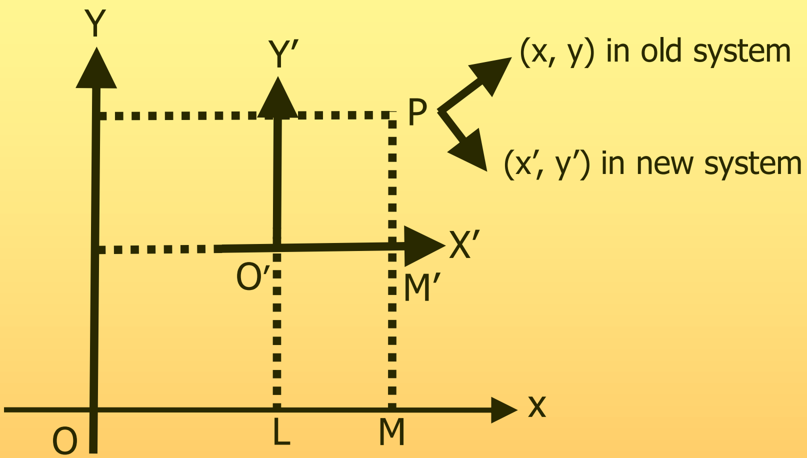
  where *k* is a *parameter.*

- **Translation of Axes:** Sometimes a problem in analytical geometry becomes simplified by shifting the origin (0, 0) to some convenient point (h, k). Then the point (h, k) is referred as the new origin O′ and the new coordinate axes (i.e., x-axis and y-axis) through O′ are drawn parallel to the corresponding old axis. Such a transformation of the rectangular coordinate system is called a translation of axes.

Consider a point P. Let (x, y) be its coordinated with regard to the old axes OX and OY.

Let O′X′ and O′Y′ be the new axes parallel to OX and OY respectively, where O′ is the new origin. Let the coordinates of O′ be (h, k) with reference to the old axes, Draw O′L ⊥ OX and PM ⊥ OX. Then, OL = h, LO′ = k, OM = x and MP = y.

Let (x′, y′) be the coordinates of P with reference to the new axes O′X′ and O′Y′. Then O′M′ = x′ and M′P = y′



Now $OM = OL + LM = OL + O'M'$

$\Rightarrow$ $x = h + x;$ ...(i)

and $MP = MM' + M'P = LO' + M'P \Rightarrow y = k + y'$ ...(ii)

Equation (i) and (ii) give the relation between the old and new coordinates. These enable us to transform the coordinated from old system to the new and vice versa.

Thank You…